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ABSTRACT

Four papers focus on the following questions and issues in research on teaching mathematics: environment, desired outcomes, learner, and implications of research for teachers and teacher education. Formal reactions follow each paper presented by individuals from an educational discipline other than mathematics education. Samples of discussions from conference sessions are also included. An overview of research on teaching mathematics served to introduce the conference. (MP)

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Conference Series No. 3

PROCEEDINGS OF THE RESEARCH-ON-TEACHING MATHEMATICS CONFERENCE

May 1-4, 1977

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Introduction

The Research-on-Teaching Mathematics Conference, conducted by the Institute for Research on Teaching May 1-4, 1977, was designed as a vehicle for mapping the field of research on mathematics teaching in the elementary grades. The conference strategy was to bring together mathematics educators and representatives from the fields of educational research, teaching, psychology, philosophy, and science education in order to identify the current status of mathematics teaching research and suggest goals and directions for future research. These deliberations were intended as a starting point for IRT researchers interested in studies of mathematics teaching.

The conference format included four invited, formal presentations. These presentations, based on prepared papers, focused on the following critical questions and issues in research on teaching mathematics -- environment (instructional setting), desired outcomes (curricular goals in terms of the individual and society), the elementary school learner, and implications of research for teachers and teacher education.

Formal reactions followed each invited address, presented by individuals from an educational discipline other than mathematics education. In addition, conference participants provided on-the-scene reactions and discussion.

The proceedings of those four days of deliberations are documented in this publication. Formal papers and reactions (some of which have been revised since the conference) are included here, as well as representative samples of the discussions. The discussions, which appear in script form, are based on transcripts of the conference sessions. Liberty has been taken to edit and condense these discussions, and to include only those comments directly related to the central debate.

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RESEARCH ON TEACHING MATHEMATICS: AN OVERVIEW

Perry E. Lanier

Abstract

In his paper and opening remarks for IRT's Research-on-Teaching Mathematics Conference, Lanier identifies the need for research on elementary school mathematics by citing the absence of any complete or systematic research effort in the literature. He then outlines the conference design, which was based upon five "elements of education" -- curriculum, learner, teacher education, environment, and teacher. Next, he describes the goals and purposes of the Institute for Research on Teaching, explaining how the information-processing theory subscribed to at IRT is being applied to four IRT studies of the teaching and learning of mathematics. In discussing research on mathematics teaching, Lanier refers to Weldon's notion of the "three kinds of things in the world" -- troubles, puzzles, and problems. According to Lanier, situations in elementary school mathematics that teachers and researchers are unsure how to handle are troubles that can be addressed through the puzzle form of information-processing psychology. Addressing the troubles in this manner, he concludes, will yield researchable problems for studies of mathematics teaching.

About the Author

Perry E. Lanier is a senior researcher and coordinator of the Mathematics Project at the Institute for Research on Teaching and a professor of elementary and special education at Michigan State University. With a Ph.D. from the University of Oklahoma, Lanier has had experience as a teacher, principal, teacher educator, and assistant director of the Laboratory School at the University of Oklahoma. In 1973, he wrote "Content and Process of Teacher Education," a chapter in New Directions in Education (No. 2), edited by Keith Goldhammer.

Research on Teaching Mathematics: An Overview[1]

Perry E. Lanier

Introduction

This conference was planned as a vehicle for critically analyzing the major areas of contemporary research on mathematics teaching and generating new areas. Deliberations within the conference sessions are expected to aid in the forming of philosophical parameters and guidelines for the investigations on mathematics teaching to be undertaken by Michigan State University's Institute for Research on Teaching.

The Need for Research on Mathematics Teaching

This paper deals primarily with elementary school mathematics issues, but my remarks and analyses may frequently be generalized to the whole of school mathematics. The elementary school label includes early, primary, and middle school. Persons responsible for elementary school mathematics include not only elementary teachers, but also those in higher education who teach mathematics content and methods courses for elementary teachers.

[1]Introductory remarks for the Research-on-Teaching Mathematics Conference, held at the Institute for Research on Teaching, May 1-4, 1977.

Such educators frequently also develop curriculum materials for elementary school learners and teachers. Also included among mathematics educators are mathematics supervisors -- employees of the school district, intermediate districts or state department, a small percentage of school administrators who have developed or maintained a specific interest in school mathematics, and a few scholars from the several disciplines who have an interest in conducting research on school mathematics.

The perception acquired by a casual inspection of the literature produced for and by these mathematics educators since World War II is perplexing. While the volume of such literature is increasing, as is its diversity (i.e., the range of issues traverses the psychological, sociological, and pedagogical as well as the mathematical), its purpose and direction remain vague; the result is unholistic and incomplete. If one could capture an "instant state" of mathematics education as reflected by the literature, it might be comparable to the predicament of sandhill cranes as they leave their Gulf of Mexico winter grounds for nesting sites in the Northwest. As the birds begin the trek, their behavior indicates unsureness about whether everyone is there that should be, why they are leaving, where they are going, who should lead, what configuration the process should take, and how such issues should be settled. We hope that this conference and its potential products will aid in directing the development of the field of mathematics education.

The Conference Design

The existence of five elements of education -- Curriculum, Learner, Teacher Education, Environment, and Teacher -- are defensible points of departure for establishing the necessary and sufficient conditions for generation of a field such as mathematics education. Diagrammatically,

the universe of such a field might be portrayed as a five-point figure.

Individually, each element will subsume a number of sub-elements. For example, Curriculum will include goals, content, and outcomes; Learners will include the elements of human development and the process of human learning; Teacher Education will subsume content and process of preparation and continuing professional development of teachers; Environment will include materials, time, and institutional supports or limiting factors inherent in the instructional setting. Teaching will include the propensities, thoughts, and actions of the teacher during planning and instruction and the primary influences on those propensities, thoughts, and actions.

With this structure in mind and the literature of mathematics education in hand (such as the periodic reports of Suydam and Weaver), my analysis of the literature seems to indicate the following: The curriculum category is replete with expositions on the goals of school mathematics, discussions of scope and sequence, and lists of objectives for mathematics content. Further, there are numerous reports of student outcomes from evaluation and research activities.

In literature dealing with learners, there is an abundance of human development material particularly of the Piagetian variety. There is, however, an obvious shortage of literature on how people (young people especially) learn various types of mathematical information. In the development domain, the research and expository literature is reasonably

well-balanced, but there is a dearth of expository literature on the learning process.

Literature on environment yields an infinite number of suggestions for "doing this tomorrow with your students, under these conditions." Similarly, there are hoards of expository articles on the value of manipulatives and games in general and the value of particular manipulatives and games. Due to a recent emphasis on the mathematics laboratory approach, there is a significant amount of research literature on the topic. Further, there are reports on studies dealing with variables of SES (socioeconomic status), ethnicity, and busing.

The amount of mathematical literature on elementary teacher education is vast. There are expositions on the appropriate amount and kind of mathematics to be taught, the value of various methods, and the importance of field experience. Much of the content literature is in the form of text materials. Many reports exist which address the concerns of continuing professional development. Ample consideration is given to the relationships among mathematical content, methods, and practice. There is research literature on the effectiveness of this or that set of program materials when used with this or that configuration.

A look at the teacher/teaching literature reveals the existence of activity in research on teacher attitudes and the mathematical understanding of teachers. It also reflects a recent, passing interest in "discovery" or Socratic teaching. Such literature often treats the student, rather than the teacher or a given teaching process, as the unit of analysis. Studies of classrooms and teachers have also shown that variance in student reading outcomes have generally failed to demonstrate similar variance in student mathematics outcomes.

Descriptive analyses of teachers planning for and instructing groups of learners in classrooms are obviously absent in mathematics literature. The NACOME report (1975, p. 68) captures the essence of this deficiency with the statement, "Appallingly little is known about teaching in any large fraction of U.S. classrooms."

Literature is but one indicator of the need for significant effort in the area of research on teaching mathematics, but it is a powerful one. As educators, we believe strongly in the importance of the teacher and the teaching process; yet we have been reluctant to tackle such complex phenomena. This conference is needed as a basis for generating new emphases and accelerating efforts in those areas of elementary mathematics education needing research.

The five areas I used to depict the field of mathematics education support the design of this conference since we are getting input from mathematics educators and reactions from educational psychologists, philosophers, researchers, teachers, teacher educators, etc. Another support for such a design comes from the results of an NCTM survey (cited in the NACOME report, 1975, p. 92) of teachers who were asked to list the topics they considered most important for inservice work. Approximately 40% of those surveyed listed the following: motivation, metrication, laboratory learning, slow learners, learning styles of students, mathematical games, development of instructional materials, problem solving, classroom use of computers, applications, and gifted learners. Teachers are clearly seeking assistance to problems that call for psychological and sociological knowledge. Content topics ranked lowest.

The Institute for Research on Teaching: An Overview

On April 1, 1976, the National Institute of Education established the Institute for Research on Teaching (IRT) at the Michigan State University College of Education. The Institute for Research on Teaching is

1. a center for research on teaching, emphasizing teachers' cognitive processes;
2. a forum for communication among researchers, teacher educators, and teachers;
3. a training program for researchers on teaching.

These three missions -- research, communication, and training -- are pursued as a part of the National Institute of Education effort to understand and improve the work of teachers and the quality of their preparation and continuing professional development.

The genesis of the Institute for Research on Teaching, the first agency of this type created by NIE, can be traced to the NIE-sponsored National Conference on Studies in Teaching, Washington, D.C., 1974. The 10 conference panels were organized around major points in the career of a teacher: the teacher's recruitment and selection (one panel), training (five panels), and utilization (one panel). In addition, a panel was formed to examine the roles of personnel in new instructional systems, and there were two panels dealing with research methodology and theory development.

The five panels organized around training were: Teaching as Human Interaction, Teaching as Behavior Analysis, Teaching as Skill Performance, Teaching as a Linguistic Process, and Teaching as Clinical Information Processing.

The Institute for Research on Teaching's primary philosophical base rests on the premises stated in the report (known as the Panel 6 Report) from the panel examining Teaching as Clinical Information Processing

(Shulman, 1975). The following excerpt from the panel's goal statement is a brief orientation to the IRT's philosophical theme.

> Statement of Goal: To develop an understanding of the mental life of teachers, a research-based conception of the cognitive processes that characterize that mental life, their antecedents, and their consequences for teaching and student performance.
>
> The cognitive processes with which Panel 6 was concerned include perception, expectancies, diagnostic judgment, prescription, and decision making. An understanding of these processes can be applied in further research on teacher selection, teacher education, and the development of technological or staffing innovations congruent with ways teachers think and feel. (p. 1)

The remainder of this paper will be devoted to reconciliation of this theme with the problems in the teaching of elementary school mathematics.

Based on the premises and outcomes of this conference and the IRT's commitment to pursue programmatic research on teaching as clinical information processing, the program area "Subject Matter Variables in Teacher Decision-Making" was set up to include investigations into the teaching of mathematics during the Institute's second operational year.

Information-Processing Models

Superimposing information-processing psychology on the teaching of elementary school mathematics brings to mind Weldon's (cited in Bruner, 1971, p.104) prescription, "There are three kinds of things in the world: There are troubles which we do not know quite how to handle; then there are puzzles with their clear conditions and unique solutions, marvelously elegant; and then there are problems -- and these we invent by finding an appropriate puzzle form to impose upon a trouble." Elementary school mathematics education clearly has its troubles; a sample set was formulated by a group of educators at the NIE-sponsored Conference on Basic Mathematical Skills and Learning, Euclid, Ohio, October 1975 (NIE, 1975, pp.28-32).

The educators named two categories of troubles and designated a third set, the "other" category. They called the first category, "Questions of Practical Urgency;" it dealt with such questions as "What do we mean by skills?" "What are the needed skills?" "How do we study the exceptionally successful teachers, students, and schools?" "Why are algorithms so hard to learn?" "What are children able to comprehend?" "How are children's concepts, processes, skills and beliefs captured?"

Given the steady clamor from the public for a return to basics and for the accountability of teachers and schools, and given also the concern for learning difficulties and attitudes, the troubles cited are clearly ones of practical urgency. However, will imposing information-processing psychology (our puzzle form) upon some of these troubles of elementary mathematics education lead to the invention of problems subject to solution?

Before we address such a question, the Euclid Conference committee's topics for study should be examined more closely. The following items are (in the committee's opinion) descriptions of the six most important "Questions of Practical Urgency."

1. What do we mean by skills? That is, (1) what do we -- and should we -- have in mind when we talk about basic mathematical skills? Some mapping of the alternative possibilities seems imperative to avoid an overly narrow notion of skills; (2) what does it mean, and require, in psychological terms, to perform in a skilled rather than an unskilled way, i.e., is it a matter of routinization, knowing what to do?

2. Study exceptionally successful teachers, students, and schools. This procedure seems clear and important -- a matter of tapping information resources already on hand. However, it may not be easy to do, especially in light of confusion concerning what constitutes being successful. However, the results of such a study could lead to the identification of teaching strategies or pedagogical methods of great value.

3. Perception of structures; noticing (relation to taste, sense of aesthetics). This asks whether and when people spontaneously notice in their out-of-math-class experience structures and occasions for tactics

learned in class. Such spontaneous noticing is crucial if there is to be good payoff outside the classroom; yet research on curriculum success typically ignores this, and curriculums are rarely designed with this transfer objective continuously in mind. In particular, what are children's tacit and/or spontaneous concepts, processes, skills, beliefs? This is fundamental. We must know where they are as well as where we want to move them before we can decide how to move them. Though many beliefs prevail concerning what attitudes and skills children do or do not have, it became clear at the conference that there is much simple descriptive work that needs to be done along with the more subtle probing of underlying cognitive processes.

4. True functional needs (remember Xerox). In the midst of controversy about what skills people ought to have, this questions calls for the matter to be studied empirically -- track people's behavior and see where they need which skills. "Remember Xerox" reminds us that "needs" should not merely mean needs exercised by current practices. As with the office copier, there may be unrecognized needs. In particular, the potential of hand calculators must be explored. What are their innovative uses? Learning uses? What would be the long range implications of substituting calculators for pencil-and-paper computation?

5. Why are algorithms so hard to learn? In particular, how do students acquire algorithmic routines from problem situations or instructional algorithms?

6. What are children able to comprehend? All sorts of claims are made concerning what children of various ages are or are not able to do, and, therefore, about what is or is not possible to teach. The claims are contradictory and need to be resolved. Also, what is the relationship between comprehension and computation?

"Questions Related to Understanding " was the second category of trouble identified at the Euclid Conference. "Troubles in Understanding" are characteristically of a greater magnitude than those of practical urgency in that they appear to be less limited to specific content mathematics; they are questions of learning -- models, problems of transfer, and techniques of practice. (For elaboration of the group's major trouble areas please refer to the Euclid Conference Report, Vol. II.)

7. What models for learning are generalizable across content areas? That is, what models of learning and learning procedures that one can deliberately practice apply indifferently to many content areas? How good are these models and can we import them to apply to problems of math education? This asks for some heavy thinking about how to apply already developed concepts to the problem at hand. One way to approach this question would be to study elementary information processes -- memory management processes, types of encodings, types of search procedures, basic symbol manipulation repertoires. In particular, the study of information processing underlying learning should be contrasted with terminal performance. This is a dis-

tinction that has not been explored carefully and might be very fruitful to explore. We dare not assume that the processing involved in a partially mastered skill bears some simple relations to the final process -- e.g., merely slower, less reliable. Rather, the step from halting and inadequate performance to mastery may involve substantial shifts in the organization of the process.

8. To what extent are concepts context-bound or general, and when? This asks about the nature and problems of transfer -- how readily will concepts mastered in one context carry over to others, and what kind of nudging, explicit pointing out of connections or the like, encourages such transfer. Everyday examples show that the help needed to achieve transfer varies enormously with the sort of material under consideration and that this question needs to be asked repeatedly if education wants to achieve maximum leverage.

9. What techniques of practice are being used and how effective are they? This asks what is known about affective practice -- different patterns, schedules, motivation -- and whether this information is in fact being used in current math programs. One suspects, with regard to learning algorithms, that there may be valuable parallels with, and information importable from, such areas as musical or athletic training, where routinization is at a premium.

Undoubtedly, the nine questions, if examined thoroughly, are sufficient to establish multiple and diverse troubles of understanding in elementary mathematics education that we do not know quite how to handle. If the nine are representative of our types of troubles, then following Weldon's prescription, we need to impose an appropriate puzzle form upon them. Since information processing psychology is concerned with how humans act upon (process) data (information), this puzzle form appears promising for those troubles related to learning and teaching mathematics (Resnick, 1976).

If information-processing psychology is a puzzle form of some promise for our "troubles" in the teaching and learning of mathematics, then an examination of its assumptions and principles and their respective applications and strengths is in order. This paper will attempt to identify those properties of information-processing theory that have import for the teaching and learning of mathematics.

A skeletal yet broad perspective of information-processing theory can be obtained by considering Herbert Simon's premises of human problem solving, Lauren Resnick's analyses for instructional purposes, and Arthur Elstein's and Lee Shulman's use of the theory in their research on medical inquiry. Such a perspective reflects the conceptual underpinnings of the theory as well as the methodologies inherent within it.

Simon's (1971) basic premise is that problems confronting humans (including, but not restricted to the teaching and learning of mathematics) are very complex and that to tackle those problems, the problem solver has to simplify his/her task, i.e., "The central process in human problem solving . . . [is using] heuristic methods to carry out highly selective searches, hence [cutting] down enormous problem spaces to sizes that a slow serial processor could handle" (p. 147). Simon deduces that the first necessary step for humans in solving a complex problem is simplification. He supports the necessity of this process by speaking about the capability of the human problem solver.

> The capacity of the human mind for formulating and solving complex problems is very small compared with the size of the problems whose solution is required for objectively rational behavior in the real world -- or even for a reasonable approximation to such objective rationality. [After proposing the principles of "bounded rationality" to describe man's limited capacity for rational thinking, Simon continues] The first consequence of the principle of bounded rationality is that the intended rationality of an actor requires him to construct a simplified model of the real situation in order to deal with it. He behaves rationally with respect to this model, and such behavior is not even approximately optimal with respect to the real world. To predict his behavior we must understand the way in which this simplified model is constructed, and its construction will certainly be related to his psychological properties as a perceiving, thinking and learning animal. (1957, p. 198)

Simon indicates his belief in the specific power of information processing and its general scope and limitations as a theory in an invited address to the mathematics education community. He spoke on the trouble of "Learning with Understanding." His belief that the information-processing form can be imposed on this trouble is captured in the following paragraph.

> In my remarks today, I should like to describe how contemporary information-processing approaches to thinking and learning are beginning to illuminate the rote-meaningful distinction, enabling us to determine with some precision what the student has stored in memory as a result of a learning experience, what the consequences are to storing one thing rather than another, and how particular kinds of learning can be encouraged and others discouraged. Our new knowledge and precision in these matters is closely linked with our growing ability to write computer programs that describe and simulate in detail the processes that humans use to carry out complex cognitive tasks, and that describe how the knowledge and information used in these tasks is stored in human memory (Simon, 1973 p. 1).

Finally, Simon's (1971) derivation of the four propositions of the theory of human problem solving illuminates the structure of the theory.* The propositions are:

1. A few, and only a few, gross characteristics of the human information-processing system are invariant over task and problem solver.

2. These characteristics are sufficient to determine that a task environment is represented as a problem space and that problem solving takes place in a problem space.

3. The structure of the task environment determines the possible structure of the problem space.

4. The structure of the problem space determines the possible programs that can be used for problem solving. (p. 149)

Lauren Resnick's work (1976) "Task Analysis in Instructional Design:

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*It is important to note that Simon's theoretical emphasis on"task environment" and "problem space" is consonant with the more recently recognized importance of "context" by those using an empirical approach to research on teaching (see Current Directions in Research on Teaching (Conf. Ser. No. 1). East Lansing, Michigan: The Institute for Research on Teaching, Michigan State University, 1977).

Some Cases from Mathematics" identifies several unique parts of the information-processing theory. Her distinction between empirical and rational analyses seems helpful in illuminating the compatibility of the theory of human problem solving to the simulated problem-solving ability of the computer.

Resnick states that "most information-processing theories and models characterize the human mind in terms of the way information is stored, accessed and operated upon" (p. 168). The functions associated with the mind's handling of information are at once together and separate. Grasping abstractions is difficult but seems to be facilitated by first considering storage mechanisms.

Information Storage

To the information-processing psychologists, there are two levels or modes of storing information. Working (short-term) memory is one level, and semantic (long-term) memory is the other. Although both are referred to as memory structures, they differ in function and description.

Working memory is the system's place for processing information acquired perceptually from the environment and reconciling that information with information called up from the long-term memory. It is also the place holder for processing new information that is to be sent to long-term memory. Simon (1976) summarizes this notion by saying, "The short-term memory is a way station not only for incoming sensory stimuli, after recognition, and out-going motor signals, but also for the symbols that are inputs and outputs of the system's information processes" (p. 93).

Two other characteristics of the working memory are important to our interests here. The working memory is assumed to process information serially, i.e., one action at a time. Second, working memory is thought

to have limited capacity, and this capacity is measured in chunks (a chunk being a familiar item). Resnick (1976) explains that working memory has a limited number of "slots" that can be filled, so that it is only by rehearsing or by "chunking" material into larger units (so that a body of interrelated information takes up a single slot) that loss of information from memory can be avoided.

The second level of storing information is semantic memory. It is the long-term memory and contains everything one knows. An interesting conjecture of this level is that the information is probably permanently stored. Furthermore, the memory is assumed to be associative, that is, to consist of a network of relations -- nodes and links. According to Simon (1976), this node-link structure and its accompanying discrimination net in long-term memory are analogous to the cross-referenced text of an encyclopedia and its accompanying index. The capacity of the semantic memory is considered to be infinite.

Resnick (1976) asserts that within this general bi-level structure, "Short-term memory is pivotal. It is only by being processed in working memory that material from the external environment can enter the individual's long-term store of knowledge, and only by entering working memory can information from the long-term store be accessed and used in the course of thinking" (p. 64). It is assumed that only milliseconds are needed to retrieve information from the long-term memory, but 5-10 seconds are necessary to process new information into the long-term structure.

One can readily see the power and utility of the working memory concept in terms of the teaching and learning of mathematics. Implications for teaching and training teachers abound. Dealing with memory issues is a pervasive trouble. The information-processing theory premise that

the human mind possesses a bi-level faculty for dealing with memory may be a useful form to impose on one of school mathematic's major troubles.

One further characteristic of information-processing activity dealing with the analyses of complex tasks should be noted. Resnick makes a distinction between rational analyses and empirical analyses of such tasks. In information-processing work, rational analyses are descriptions of ideal programs. They represent humans' "best speculations" and are simulated on computers for testing against the empirical. One might say rational analyses provide the "baseline" for comparison with what is actually done in real situations.

Empirical analyses, on the other hand, are drawn from a real situation; that is, they are based on the interpretation of data from actual human performance. In practice, the rational analyses may be the point of departure for empirical data collection. As a result, the two are rarely sharply separated. The value and potential of rational analyses in studying processes underlying human performance, including the contribution of computer simulation, is discussed by Simon.

> The limits of process analysis have been fixed by the limits on our conceptual apparatus for describing process, and the limits of our ability to devise empirical methods of observing and dissecting process. The development of computer simulation techniques has relaxed these limits considerably. (1976, p. 97).

Elstein and Shulman (1976) confirm that the computer program (rational analysis) is almost an indigenous characteristic of information-processing psychology. They state, "Writing a computer program as a test of the sufficiency with which a particular sequence has been traced is a trademark of the contemporary information-processing approach" (p. 28).

The report of Elstein and Shulman's (1976) investigations in medical inquiry, which were based on the canons of information-processing psychology, is a resource rich in methodology identification and description. They artfully capture the common elements of information-processing investigation in the following statement:

> Methodologically, information-processing research generally relies on introspective reports to determine thought processes, heuristics, symbolic manipulations, or decision rules needed to solve a particular problem. These are formalized by the investigator into a computer program or other model that should perform the task much as the human did, if the detailed specifications are adequate. A major feature of many of the situations studied with this approach...is that the starting situation and desired end are both given (p. 28).

Further, they describe and contrast the two major approaches that have been used to research the intellectual processes of humans. The first is the "process-tracing" approach which attempts to describe the intellectual processes used by subjects as they render judgments, make decisions, or solve problems. In this approach, human thinking and problem solving are typically viewed as a series of operations that are susceptible to either verbal characterization or restatement in a computer simulation program.

"Black-box" investigations, the second approach, attempt to "model the processing of the judge mathematically through studies of input-output relations." This tradition, which has contributed to the field of judgment theory, consists of two primary approaches, the Lens model and the Bayesian model (Elstein & Shulman, 1976, p. 15). (The interested researcher may wish to peruse other sections of the Elstein-Shulman report for in-depth coverage of methodology or Chapter 1 of the 1975 Review of Research in Education for a comprehensive summary of the project.)

In sum, the IRT's research projects on the teaching of mathematics will impose the forms of information-processing psychology on the questions selected for investigation. Further, the research paradigms will reflect the traditions of process-tracing studies or the study of input-output relations. The investigations will also be characterized by the premise that teaching is one of many classes of problems, i.e., teaching is a problem-solving process. This view of teaching (which has dominated one of our developmental programs in teacher education here at MSU) and information-processing psychology do, indeed, seem complementary.

The nature and character of the IRT studies in math will focus on the mental life of the teacher. This is in contrast to focusing only on the behavior of the teacher, which is assumed to be a consequence of thought. We do not disparage studies of teacher behavior, but the IRT's interest in behavior is as a function of the presence, structure, or even absence of thought.

Our commitment to a focus on the mental life of teachers is reinforced by Easley (1975) in his assertion that, "Learning theorists in education have... been attempting to design functional systems which teachers could be taught how to operate. Unfortunately, these designs of instructional systems have never taken into account the conceptual systems of the teachers and students. Putting it in Skinnerian theory, the full set of operants were never observed and, therefore, were not taken into account in the programming" (p. 36).

To prevent Institute studies from considering less than "the full set of operants," the IRT math investigations will also be undertaken from an ecological perspective. Such a perspective not only recognizes the conceptual system of teachers and learners but the "groupness" (i.e., the multiplicity of varied sets of individuals) quality of a class-

room. Barr and Dreeben (in press) in a review of research on instruction in classrooms which contrasts the "classroom instruction tradition" and the "school resources and achievement tradition" paradigms, emphatically point out that "classes are social aggregates, [that] have social, temporal, spatial, and technological properties."

One set of studies will likely center around the successful teacher or a teacher's periods of success. Our strategy would be in the process-tracing tradition and would probably, via a participant/observer plan, produce a systematic and comprehensive log of classroom transactions over a designated period of time. Each observation would be followed by a stimulated recall session with the teacher. After organizing and studying the data, we could identify episodes judged successful by the teacher and by the observer, and if permissible, by other significant individuals or groups. With these episodes in hand, we would attempt to identify the chunks in the teacher's working memory at the time in order to acquire an inferential glimpse of that teacher's task simplification scheme for those episodes.

While there are other questions we could ask about the chunks, the important observation is our definite concern for generalizability and theory development. As Easley (1975) has said, "The quest for theory and generalizability through theoretical understanding has supreme value for the practical educator" (p. 37).

The proposed studies should be able to build on the results of an earlier study by IRT researcher Janet Shroyer. By observing and videotaping three upper elementary teachers teaching mathematics in their classrooms, Shroyer attempted to capture the influence of goal specificity on the activities selected by the teacher for the students.

Shroyer's study should also serve as a prelude to a second set of studies where an effort will be made to capture the thought processes of both teacher and student. Again, process tracing, where introspection is the primary source of documentation, would be an appropriate research methodology. Technically, we would be engaged in a simultaneous stimulated recall process.

Easley (1975) also discusses the potential of parallel processing for capturing the social interaction of teacher and pupil. Bauersfeld (in press) speaks of the lack of congruence between "matter meant," "matter taught," and "matter learned." Research on such congruence would include questions of what, how, and why the learner is thinking as he or she is, and how the learner's thinking is related to the teacher's thinking. Such questions of congruence are seen as especially promising. Shulman and Shroyer (1976) have noted:

> The current movement toward applying...information-processing [psychology] to the phenomena of teaching, as it has been applied to the processes of learning, perhaps presage an eventual, though still remote, unified information-processing theory of teaching and learning (p. 65).

We are thus seriously considering embarking upon a set of studies on the concurrent thought processes of teachers and their students.

A third set of studies also focusing on the mental life of students, as well as that of the teacher, could be conducted with an experimental component included. These studies would be based on the assumptions that:

1. Successful teaching of elementary school mathematics requires a blending of mathematical, psychological, and sociological knowledge and theory.

2. Present teaching practice reveals little actual or intuitive awareness of the principles of information-processing psychology.

3. An intervention designed to formalize a teacher's understanding and the potential application of information-processing psychology will alter the teacher's thought processes regarding the teaching of mathematics to elementary children.

A paradigm with this experimental component would not necessitate abandoning the field study or natural setting approach that is presently the most lucrative means of studying teaching. Bronfenbrenner (1976) has spoken of the advantages of combining aspects of the field and experimental methodologies.

Another example of a type of study the Institute might initiate is one that looks at a teacher's judgments regarding students' achievement. The field of attribution theory could be imposed on the achievement phenomena, i.e.,the cause of a given instructional outcome can be attributed to the difficulty of the task, the aptitude of the learner, the effort of the learner, luck, performance of the teacher, and the type of school. The Lens model is an appropriate mechanism to use in this type of study. In particular, cues from each of the six attributes can be collected and the influence of each on the teacher's judgment can be determined. Further, the teacher's judgment can be compared with the environment (or reality). Hammond (1975) and others have shown that feedback on an accumulation of judgments by a teacher that differ significantly with reality is a powerful influence in improving judgments.

Questions that could be addressed in studies of this type are considerations of judgments and the implications for subsequent instructional goals decisions, and strategy selections which would include activity and management decisions.

Summary

Discussed above are four examples of studies that are compatible with the Institute's theme of Teaching as Clinical Information Processing, and its proposed slogan, "In teaching, it is the thought that counts."

Throughout the conference we expect numerous other compatible and worthy examples will become apparent as we examine the trouble in the areas of school mathematics curriculum, learning, environment, and teacher education.

In conclusion, and based on the literature of elementary mathematics education, research on the teaching of mathematics is an area in need of analysis. Such an analysis would be incomplete without a multi-disciplinary perspective; i.e., mathematical, psychological, and sociological views are necessary. Some of the things in elementary school mathematics that we do not quite know how to handle (the troubles) can be addressed via information-processing psychology (a promising puzzle form). Addressing the troubles in such a manner might well yield problems that are subject to solution.

I believe this conference gives us the opportunity to analyze and synthesize research in new and different ways. We are at a new place in educational research; we have new information, new technology, new and old troubles, and new as well as familiar colleagues. James Greeno (1976) states, "I think we can assert confidently that our state of knowledge and understanding in cognitive psychology has now developed to the point where meaningful contact can be made with the content of instruction" (p. 158).

Simon (1976) says that what has occurred thus far in information-processing work "should be regarded as hints of the potential of powerful new tools, not as predictions of what these tools will discover for us when we have learned to exploit them effectively" (p. 97).

To put our hopes for research in general and this conference in particular in perspective, let me paraphrase a comment made recently by Lee Shulman. Research on teaching mathematics will not produce results

that simplify teaching, but results that will bring order and understanding to its complexity.

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CURRICULAR IMPLICATIONS FOR RESEARCH ON TEACHING MATHEMATICS

Robert B. Davis

Abstract

Davis asserts that differentiation is essential. Researchers need taxonomic and demographic breakdowns of kinds of knowledge and skills, kinds of students, their personal goals, intentions, vocational and social needs, and kinds of learning experience. Researchers also need to take advantage of practitioner knowledge, to seek a theory in local professional terms and develop a model of relevant phenomena so as to be able to reason (and, if appropriate, count) within this model. Discussed in this paper are twenty-three kinds of mathematical knowledge of mathematical thinking.

About the Author

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CURRICULAR IMPLICATIONS FOR RESEARCH ON TEACHING MATHEMATICS

Robert B. Davis

> The truth knocks on the door and you say, "Go away, I'm looking for the truth," and so it goes away.
>
> -- Robert M. Pirsig

The work being undertaken at the Institute for Research on Teaching offers a rare opportunity, inasmuch as it allows study of the interaction between the king of thing to be learned and the way in which it is taught. In the midst of the intricacies of contemporary scholarship, we must not lose sight of common sense, and common sense surely tells us that differentiation of methods and goals is essential. The most familiar forms of studying teaching have looked for phenomena that were not dependent upon specific content to be learned, but on the contrary, produced more (or less) learning over a wide range of content.

But common sense tells us this familiar approach must miss as much truth as it reveals. There are content areas where ever-greater precision is important, and where precision may be counter-productive. There are content areas (and groups of students) in which teachers who are very timid may also be very effective, but there are also student groups and content areas where timid teachers are not so effective.

To study the effectiveness of teaching, we need — ideally — taxonomic and demographic breakdowns for five areas:

1. Kinds of knowledge or skills to be acquired
2. Kinds of students
3. Kinds of learning experiences
4. Kinds of student personal goals or intents
5. Future vocational or social needs

Ideally, one should only speak of effective teaching for specific combinations of these variables: *this* specific kind of content, to be learned by *this* kind of student, whose present orientation or intent is thus-and-such. In the usual studies of teaching, all of these variables and their interactions have been provided for by the artistry of the teacher, while the "scientific" study of teaching has looked for universal truths that apply to all content areas, all students, and all contexts. By so doing, the "scientific" study of teaching limits itself to a domain that is even more severely restricted than that of exobiology.

Contrast, for a moment, teaching with the operation of a hospital. In a good hospital there is differentiation everywhere. Patients who are allergic to penicillin are identified, and this most common of antibiotics is never administered to them. Blood types are identified, and the distinction is respected. Even at gross macroscopic levels, a maternity case is distinguished from a broken leg or infectious hepatitis, and appropriate treatments are devised separately and studied separately. There may be some common attributes shared by all effective physicians, but they recede into the background, and it is the rare dermatologist who delivers babies, the rare psychiatrist who enters the operating room to work as anesthesiologist. Differentiation is everywhere.

Explication of Practitioner Knowledge

Before considering curriculum, we need a brief digression on methodology. David Hawkins has called attention to one of the most important — and most neglected — aspects of science: the analysis and explication of practitioner knowledge. Practitioners knew how to make beer, wine, and glass, cure leather, make and use maps and violins, build cathedrals, and control pain and fever long before any theorists understood these processes. Pitchers could throw curve balls before physicists could explain them — or even believe in their possibility.

Practitioners do know something, and a major stage in science occurs when theorists study practitioner knowledge and try to relate it to theoretical modes of thought. This stage has been bypassed in education. Teaching is studied with the implicit assumption that teachers do not know what they are doing and theorists must come in and tell them. Yet, paradoxically, we do believe in teaching, but in the most mystical way imaginable; David Page's great successes with students have often been dismissed with the remark, "Why, of course — David Page can teach anything to anyone! What does that prove?" The remark itself proves something.

I believe that one of the major tasks in the study of teaching is the identification, analysis, and explication of teacher knowledge — the expertise of the most successful practitioners. This will prove a very demanding task because it is not easy to build bridges that connect the rhetoric of practitioners with the rhetoric of science. The difficulty lies not in the relatively trivial matter of teachers' using one set of words and theorist's using another; it is more fundamental than that. The following is an example — an incident so far out that I have never related it before.

One semester when I was teaching at the Massachusetts Institute of Technology,

I was asked to teach a special section of students who were taking Math 12, integral calculus, for the second time because they had previously failed it. After the first test one boy came to see me in my office. He had received a very low grade, and, in fact, his test paper showed almost no success in attacking any of the problems. He asked me if his paper wasn't worth a higher grade than I had given it. Clearly, it was not; you couldn't find three connected lines of work anywhere on the paper, and hardly any writing at all, for that matter. Discussion revealed that he had failed Math 12 not once, but twice. This was his third attempt to pass the course, and all of his future studies depended on it.

As I looked at that boy's facial expression, and as I heard the inflections in his voice and his precise choice of words, I thought of something I had read while waiting in a dentist's office, in a source that is hardly ever cited in the professional literature: the Reader's Digest. The article had, indeed, dealt with grading papers. An experienced and respected older teacher had given a passing grade to a very poor student. A friend, visiting, had commented that the student's work did not deserve a passing grade. "It certainly does not, but by the end of the year it will," the teacher had replied. With that anecdote in mind I carefully went over my own student's test paper, while he and I discussed it. I used every possible reason for giving partial credit for anything that I could find, any step that was right or partially right, or that could be said to contain the germ of an idea, etc. When we were done, his grade had been raised very substantially — to about 65 or 70 percent — and he was obviously very much impressed.

The reason I have never forgotten this episode is that, at the end of the semester, that student, whose history (and first test paper) could hardly have been more dismal, was the highest ranking student in the class, earning

an entirely bona fide A (which M.I.T. called an "H"). Of course I don't know in scientific terms what happened. The student ended his long string of failures in math and was thereafter successful. That is a fact, and just about the only fact that one can find. Many different things could have happened, and I don't know which one did.

In that first encounter over his test grade, I had imagined that the student identifying himself with his test paper, and saying, "Aren't I worth more than that?" So I imagined that my best role as a teacher was a two-pronged strategy: go over the work and interpret it so as to show him that he *was* "worth more than that," while at the same time asserting the integrity of the content, the objectivity, and the superiority of correct reasoning over confusion. Mathematics is not a "do-your-own-thing" kind of subject.

That's what I imagined I was doing; what the student thought about it, of course, I don't know.

I am also uncertain about what caused the change in the student's performance. Maybe he had previously lacked confidence, and now he no longer did. Maybe he had previously concluded that he never got a fair hearing from a math teacher, and his fearful anticipation of an inevitable rejection may have prevented him from performing well. Maybe he'd just never really gotten it all together before, and maybe he'd suddenly seen the handwriting on the wall; maybe, for the first time, he saw his choices more clearly: he was going to fail (again!) if he didn't really get down to business, and he didn't have to fail, he could pass, provided he did get down to business.

I can think of several dozen other possible explanations (e.g.,maybe he knew he had conned me, and he worked harder to overcome a sense of guilt), but at this time there is probably no way to choose among them. The point is, this is the kind of data a practitioner has, this is the kind of decision he must make, and this is the kind of result he can observe.

How is such practitioner knowledge different from the more common kinds of scientific knowledge?

First, there was a large amount of input data. I saw the boy's posture, I saw his facial expression (and how it changed as we talked); I heard the overtones in his inflection; I heard his choice of words, his pauses and hesitations; I saw his shifting glances, and so on — a very large amount of input data, indeed.

Second, I was not consciously aware of which data I attended to, nor why I evaluated it as I did.

Third — though this is more similar to science than it is different — the practitioner is confronted with a large range of possible theories as to what really went on, with no evident way to select one theory in preference to the others.

Fourth, there is no evident invariance here. I emphatically do NOT suggest that I would have doen the same thing with every student; I would not have. Indeed, most of my test grading is as objective as I can make it. For some unknown reason, on that single occasion, with that specific student, I had the sense that I could best help his studies and growth in the long run by re-grading that one single test. When would the next case like this come along? Who knows? Maybe never. (I have been teaching for over 30 years, and I can't think of another case that has ever come along that seemed to me to be essentially similar to this one.)

Fifth, what did the teacher do? If there is a vast input of data from the student to the teacher, there is also a vast outpouring of data from the teacher that becomes an input into the student's system. I do not know which aspects of what I did were important to that student, nor the extent of their import.

Sixth — and this is a very major difference, indeed — as a professional I was, at the moment, trying to serve one single client. I was not concerned with what is most often helpful to most people. The percent of time that Treatment X succeeds was NOT my concern. I wanted a treatment that would work *right now, for this one single student.*

The sixth point is so fundamental that it demands further discussion. Educational research in the United States has (perhaps unconsciously) become administration-centered. It deals with the kinds of decisions that administrators make, the kinds of data that administrators have, the selection among options that administrators have available, and the kinds of outcomes about which administrators can be aware. This is very different from the data, decisions, and options that confront a teacher.

A policy of giving every student at least 65 on his first test of the semester is something an administrator might conceivably hope to implement. A policy of changing a single student's grade when it seems to be a good idea is not. A teacher can think in those terms, but an administrator cannot. A teacher receives this vast amount of input data from students and broadcasts to students a similarly vast amount of data — from posture to punctuality, from priorities to pronunciation, from praise to put-downs. A teacher can — and, I believe, should — deal in individual cases: this single event with this single student.

My son receives an insulin injection every morning. For him, it is literally a vital necessity. For all the other members of our family it would be harmful and might even be fatal. My son's insulin dosage is also a single event. The 45 units he received last week may need to be changed. By constant monitoring of the chemical composition of his urine it is possible to adjust the insulin dosage to his present needs.

In educational research we have lost the ability to think in terms of individual cases. There has been a harmful symbiosis of administrative knowledge types, statistical methodology, and seemingly scientific generalizations. What is best for *most* people may not be best for *you*. A good teacher tries to make these individual adjustments, but research methodology tries to cast them aside. The first instinct of research has been to count — that is, to count numbers of cases or numbers of people. This is not an inevitable limitation of science. In determining insulin dosage for my son, who cares what the average human dosage is? Who cares how many people in the state of Illinois needed 45 units this morning? There *is* counting to be done, *but it is counting within a model of the relevant mechanism;* one counts, for example, the number of units of insulin, and one measures the amounts of certain chemicals in the urine. One counts within the detailed model of the relevant mechanism (here, sugar-insulin balance and metabolic chemistry), and one deals with individual events and individual people.

Educational research deals, for example, with average class size. The correct local professional decision deals with questions such as: How large a class can Mr. Wilson handle? Can we have Brian and Harold in the same class? How large can a class be if John, Ian, Willie, and Hero must all be in it? How large a class will fit into Room 38 during the hot weather?

Comparing administrative knowledge with local professional decision-making, one finds that they differ in the kind and amount of input information, the kind and amount of output information, and the specificity or generality of focus of concern. Bridging these gaps is not a trivial matter — indeed, "bridging these gaps" may well be the wrong way to think about the problem. What we need is a theory of local professional knowledge, not cast in generalized administrative terminology.

Ground Rules for Discussion of the Curriculum

For our consideration of the curriculum, then, we have these ground rules:

1. Differentiation is necessary: this specific content, for this specific student, who has this specific kind of approach, etc.
2. Research must build on the knowledge of teachers and practitioner belief systems.
3. Theory should be sought in local professional terms, not in the more remote, abstract, generalized, administrative terms.
4. A model of the relevant phenomena must be built so that one can reason (and, where appropriate, count)within this model, instead of merely counting instances or frequencies of general cases.

Kinds of Mathematical Knowledge or Mathematical Thinking

It is not true that math is math is math. Different kinds of knowledge and different kinds of tasks appear within the domain of mathematics.

For several years, my colleagues* at the Laboratory School of the University of Illinois and I have been trying to distinguish among different kinds of knowledge (or different kinds of information processing) that are involved in learning and using arithmetic, algebra, and analytic geometry. This quest has not reached its goal, but it is worth reporting. To comprehend it, imagine chemists before Mendeleyev and perhaps even before Lavoisier. It was clear that there were lots of different chemical substances and that both similarities and differences had to be understood, but at first, one had only the large disorderly array of the substances themselves — a black powder here,

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*Curtis McKnight, Stanley Erlwanger, Jim Raths, Elizabeth Jockusch, and Katie Reynolds Hannibal, among others.

an orange one there (and there may have been doubts and disagreements even at this level, since the orange powder came to be known as "red mercuric oxide"). Gradually, chemists came to perceive the order through this disarray, through Lavoisier's notion of elements and compounds, Mendeleyev's "periodic table," etc.

This ordering is the kind of task we are undertaking. Whether we as yet have found any of the true lines of demarcation remains unanswered. Our present list of more-or-less distinguishable kinds of mental information processing includes the following:

"Unit" sub-procedures. The students we study are at least in third grade, and more often at least 11 years old. Hence, there are many things they can do, though not necessarily perfectly. They can add 3 + 5, or multiply 3 X 5, and so on. These skills were acquired in earlier grades and now have something of an "automatic" character about them; students usually know the answer without any conscious awareness of how they solved the problem. I call these *unit sub-procedures* -- "unit" because of their relatively indecomposable gestalt, and "sub-procedures" because (as in computer programming) they will usually be combined into larger aggregations in order to solve some problem.

New S-algorithmic procedures. This is one of the new things students are trying to learn in our classes. Indeed, some people imagine this represents all of mathematics. An S-algorithmic procedure is a sequence of unit sub-procedures, where the N^{th} sub-procedure in the sequence is called up by some stimulus, very often visual, that is available upon the completion of the $(N-1)^{st}$ step. The first step is the original problem, and the final step is its solution. We presently use this category mainly for algorithms that are learned by rote. For example, an S-algorithmic solution of this equation:

$$\frac{3}{x} = \frac{6}{3x - 5}$$

would, following Stein and Crabill's recommended procedure, go as follows:

$$\frac{3}{x} = \frac{6}{3x - 5}$$

$$\downarrow \quad \cdot (x)(3x - 5)$$

$$\frac{3}{x} \cdot (x)(3x - 5) = \frac{6}{3x - 5} \cdot (x)(3x - 5)$$

$$\downarrow \quad \text{following rule for multiplying fractions.}$$

$$\frac{3}{x} \cdot \frac{x(3x - 5)}{1} = \frac{6}{3x - 5} \cdot \frac{x(3x - 5)}{1}$$

$$\downarrow \quad \text{multiplying}$$

$$\frac{3x(3x - 5)}{x} = \frac{6x(3x - 5)}{3x - 5}$$

$$\downarrow \quad \text{cancelling}$$

$$\frac{3\cancel{x}^{1}(3x - 5)}{\cancel{x}_{1}} = \frac{6x\cancel{(3x - 5)}^{1}}{\cancel{3x - 5}_{1}}$$

$$\downarrow \quad \text{simplifying}$$

$$3(3x - 5) = 6x$$

$$\downarrow \quad \text{multiplying out}$$

$$9x - 15 = 6x$$

$$\downarrow \quad - 6x$$

$$9x - 15 - 6x = 6x - 6x$$

$$\downarrow \quad \text{collecting like terms}$$

$$3x - 15 = 0$$

$$\downarrow \quad + 15$$

$$3x = 15$$

$$\downarrow \quad \div 3$$

$$x = 5 \ .$$

Some parents and some students believe that all of mathematics should be handled on this basis. Personally, I doubt that this is possible. The controversy, however, is a very important one. I believe it underlies much of the story of the "new math" revolution of the 1960's. Many creators of new math curricula believed that one could not use this approach of explicit rote teaching of S-algorithms (not, of course, the language they would have chosen!) to create a royal road to all of mathematics; on the other hand, many conservative opponents of new math believed that one could, and should, teach all of mathematics by explicit,rote S-algorithmic procedures (not the language they would have chosen, either!). Hence, much of the argument over "discovery learning" was rather beside the point. If you believe that an explicit algorithmic presentation is possible, you can easily decide that it is desirable. The following are some of the arguments against this approach.

1. There is evidence from Guy Groen and his colleagues that people think about mathematics in a way that is essentially different from the way they were taught to think about mathematics. Resnick (correctly, in our view) uses this as an argument for the following criterion of teaching: Teachers must teach in such a way that they do no
2. If one learns only these explicit S-algorithmic procedures, s/he will inevitably be confronted in the future by problems to which they don't quite apply, and s/he will not have had any experience in adjusting and revising methods and won't be prepared to solve them.
3. All successful learning requires flexible problem solving. Nothing is really "just a matter of inflexible algorithms." Consider, for example, the difficulties in programming computers to read ordinary handwriting; even reading single letters goes beyond the simplest

explicit rote algorithms e.g., the Papert-Minsky example of reading

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4. If one trys to fight his/her way through mathematics by using only explicit S-algorithms, s/he will inevitably grind to a halt sooner or later (more likely sooner than later), because the accumulated burden of myriad unrelated rules will exceed his/her information-processing capabilities.

A great deal of progress could be made if there was some reliable resolution of this controversy. Is mathematics teaching and learning realistically reducible to nothing but rote S-algorithmic procedures?

"Conceptual" problem analysis. (This category probably requires a better formulation, but until someone creates one, the present version will do.) At each line in the preceding S-algorithmic approach, the student looks at what s/he has written and is guided, in a somewhat stimulus-response (S-R) fashion, from that visual input to the selection of the correct next step. By contrast, I can imagine another, better way of thinking about this problem. It is based on what might be called conceptual reasoning, and goes somewhat like this:

(1) $\frac{3}{x} = \frac{6}{3x - 5}$

What is it? Oh, it's an *equation*. No, an equation is really a *question;* it's asking me, "what *numbers* can I use as replacements for the variable x, so as to make equation (1) become a true statement?"

Well, equation (1) is sort of messy. Why? It has all those fractions.

This is use of the heuristic principle: find our what's making it hard...

Good. Let's get rid of those fractions!

...and try to eliminate it.

Is there any way to get rid of a fraction? Yes, a fraction, when multiplied by it's denominator, ceases to be a fraction:

Check strategic possibilities

$\frac{A}{B} \times B = A$

O.K. So — we'd *like* to multiply $\frac{3}{x}$ by x. But we can't just go ahead and do that! First, we have to find a legal way to do it. Is there a legal way to multiply $\frac{3}{x}$ by x?

Yes, because $\frac{3}{x}$ is one side of an equation, and you can multiply one side of an equation by something non-zero, provided you multiply the other side by the same thing. The important point is that you won't change the truth set of the open sentence.

The reader can probably see how this sequence could continue.

Our example introduces yet another idea, that of *heuristic analysis*, rather in the style of Polya. We do not have a clear line of demarcation between heuristic and conceptual thought; we have some examples of conceptual thought more-or-less uncontaminated by heuristics (presented below), but we have no examples of heuristic thought uncontaminated by "concepts."

What is meant by conceptual? Well, we consider the idea that *you can change an equation without changing its truth set* to be a concept. In interviews with students it is usually entirely possible to distinguish an S-algorithmic performance from a concept-based performance. If a student is carrying out an S-algorithmic solution and is asked, "Why did you just multiply by x?", s/he will answer something to the effect that s/he did that, "because that's what you do there," or "because that's what you're supposed to do." What else can be said? The recognizable visual cue is seen in an essentially S-R fashion, and s/he has produced the response that has been taught.

By contrast, if a student is giving a concept-based peroformance and is asked why s/he multiplied, s/he will answer that s/he multiplied $\frac{3}{x}$ by x to get rid of the fraction, and the other side of the equation was multiplied by x in order not to change the truth set.

Consider the last line, $x = 5$ (or, perhaps, $3x = 15$). Suppose you say to a student, "Great! Now you've found the truth set for the equation $3x = 15$, but that's the right answer to the wrong problem! You were supposed to find the true set for the equation

$$\frac{3}{x} = \frac{6}{3x - 5} \quad ."$$

The student operating in algorithmic mode doesn't know what to say, beyond something like, "Well, I did what you're supposed to do..." The student in

conceptual mode can say something like, "Well, at each step I've been careful not to change the truth set. Consequently, the truth set of equation (1) is the same as the truth set of equation (2), and that's the same as the truth set of equation (3), and so on, so the truth set of the last equation must be the same as the truth set of the original equation, so I *have* found the truth set of the original equation." (This discussion, of course, simplifies both student language and certain mathematical complexities, such as the possibility that x=0, etc. The changes are not central here.)

We have experienced great difficulty in getting most students to think about these problems conceptually. They much prefer the algorithmic approach. Perhaps this is a residue of a bad teaching approach in earlier grades that has produced an addiction to S-algorithmic methods for solving problems. On the other hand, however, a few students find the conceptual approach natural and easy. This is one major difference among students that needs to be better understood.

Perhaps it is wrong to think of the conceptual approach as an alternative to the algorithmic approach. Perhaps it is an addition to the algorithmic approach.

The following is a "pure" conceptual solution to this same problem, unsullied by any recognizable heuristics:

A seventh-grade boy solved

$$\frac{3}{x} = \frac{6}{3x - 5}$$

by giving the answer "x must be 5" almost instantly upon seeing the equation, and then explained as follows (his actual words are paraphrased):

> If you think of going from the left hand fraction to the right hand fraction, the numerator is doubled, so (to preserve equality) the denominator must be doubled. But changing x to 3x means adding one x too many; this must be exactly compensated for by subtracting 5. But if subtracting 5 exactly compensates for adding one x too many, then x must be 5.

This does not in the least resemble what he had been taught to do (Davis & McKnight, 1976, p. 274).

It may be worth commenting that the preceding occurred a year ago. Within the past month we have recorded (mostly on audio-tape) four other children in other classes who solved this problem in essentially the same way. The one redeeming feature of our studies thus far may be replicability. We are finding patterns of student behavior that can be found a year later in new students.

The structure of forms. We have become eager for our students to realize that, whereas "similarity" and "metaphor" in lay usuage are approximate, the matching of forms in mathematics is exceedingly precise. To help them achieve this realization, we begin our seventh-grade course with precise derivations of simple algebraic theorems, using what we call "U.V." and "P.N."

We want students to realize, for example, that in using

$$\frac{a}{b} = \frac{ac}{bc} ,$$

it is essential that c multiply all the rest of the numerator, and also multiply all the rest of the denominator. Developing skill in analyzing forms of this type is quite difficult for some students and quite easy for others. Some students continue to make errors of the form

$$\frac{3x^2 + 2x^2 - y}{x^2 - x^2 + 7} = \frac{3x^2 + 2 - y}{x^2 + 7} ,$$

and have great difficulty in recognizing that this cannot be justified as an instance of

$$\frac{ac}{bc} = \frac{a}{b} ,$$

because the forms cannot be made to match.

In terms of information processing, this would be merely an instance of some sort of pattern recognition. This goes beyond our present understanding,

however. In the meantime, this problem of insight into the structure of forms is so fundamental that we list it here by itself.

Specific ad hoc heuristics. Again, the basic "patterns" in mathematics are not vague resemblances, or approximate generalities. They are very precise, and are expressed by replacement and substitution ("U.V." and "P.N."). To help students develop an early familiarity with this unusual kind of precision, we begin our seventh-grade course by having students construct careful original proofs of algebraic theorems, such as

$$A + A = 2 \times A$$

$$A + [B + (C + D)] = [(A + B) + C] + D.$$

This is one of the most unusual features of our course — and, perhaps because we have been working on it for over 10 years, it is one of the most successful. We place great emphasis on some quite specific heuristics, to help students create an appropriate proof. Suppose, for example, that we have the step

$$A + A = A + A$$

and we are trying to prove

$$A + A = 2 \times A \ .$$

The heuristic analysis might go something like this:

Is either side O.K.?	Yes, the left. We needn't change it.
On the right-hand side, what is conspicuous by its *absence?*	Ans: the 2.
Where could a "2" come from?	Only from $1 + 1 = 2$
Ah! Then, where could those 1's come from?	Only from $\square \times 1 = \square$

What else needs to be changed on the right-hand side?	The *addition* sign must be replaced by a *multiplication* sign.
Which axiom changes *sums* into *products* (or vice versa)?	Only one — the distributive law.

This heuristic analysis is learned by the beginning seventh graders in a couple of weeks, to a level of considerable skill. (A typical lesson is shown, just as it took place, on the film entitled Mathematics for Gifted Children - I.) This is in sharp contrast to our experience with more general heuristics, as we see in the next section.

General all-purpose heuristics. The surprising success of the unit on specific ad hoc heuristics (previous section) led us to try some more general heuristics, such as these:

1. Try to break the problem up into parts, and work on the parts separately.
2. Try to identify the difficulties, and see if you can eliminate them.
3. Find something you *can* do, and do it.
4. Try an easier similar problem and see what you can learn from it.

Students were quite resistant to these heuristics, preferring instead to be taught explicit S-algorithmic approaches. These heuristics were developed in September, but it was February before many students were using them.

Example: Simplify

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + 1}}}}$$

There was one thing they could do here — namely, add $1 + 1$ in the denominator of the "last" fraction. It was easy to continue from there.

Verbal knowledge. The "chaining together" of a sequence of steps, as in an S-algorithmic procedure, is presumably one kind of knowledge; in some ways, it is easy to understand because it fits into an S-R bond theory, it resembles the sequence of steps in a low-level computer program, and so on. Verbal knowledge — information coded in a natural language statement — is presumably a quite different kind of knowledge. The following are examples of verbal knowledge:

1. Jennifer, a third-grade student, claimed that

$$8$$

$$\text{and} \quad 80$$

$$800$$

were all the same, because "zero means nothing," "adding zero doesn't change anything."

2. Several seventh-grade students subtracted

$$2x^2 - x^2$$

to get the answer 2, reading the problem as "two x squared, take away x squared, leaves two."

3. A geometrical way to multiply two complex numbers: multiply the absolute values, add the angles.

4. "To add numbers of unlike sign, subtract the smaller from the larger. and use the sign of the larger."

In the first two examples above, verbal statements are misapplied, causing students to get wrong answers. The third example is a verbal formula

used by our ninth-grade students to solve problems such as:

a) simplify $i^{(1/3)}$

b) simplify $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{1977}$

The fourth example has been used by Beberman; it was a common rule in "old math" textbooks, and thousands of students memorized it. It does not lead to correct answers, yet these students *got* correct answers, and they never doubted the rule.

Obviously, verbal knowledge can be dangerous. It may also be essential. In discussing

$$\frac{ac}{bc} = \frac{a}{b}$$

as a justification for "cancelling," we interpret the structure verbally to our students: "Notice that the c must multiply all the rest of the numerator..." This seems to help (though repeated treatments are often required), but perhaps it is slightly more different; it uses natural language to help the student to re-shape a visual perception.

Super-procedures. If a problem solution consists of a "super-procedure" that inputs the problem statement (probably a visual notation) and outputs a sequence of "sub-procedures" to solve the problem, our analyses of data from Erlwanger and others show that nearly all erros made by a group of fifth and sixth graders were errors in the super-procedure, not errors in the sub-procedures. Here are some examples:

1. A student converted 3/11 to a decimal as 1.4, and 2/11 to a decimal as 1.3. He was perfectly systematic about this (e.g., 3/15 became 1.8, and so on).

2. A student added .3 + .4 to get .07 .

The sub-procedure that inputs 3 and 4 and outputs 7 worked correctly (and was an appropriate choice for use here). The sub-procedure that inputs "one decimal place" and "one decimal place" and outputs "two decimal places" is a correct procedure, and worked correctly, but should not have been selected for use on this problem.

Note in both cases the visual similarity of the visual stimuli:

$$\frac{3}{15} \text{ vs. } \frac{3}{15}$$

and

$$.3 + .4 \text{ vs. } .3 \times .4 \ .$$

This seems to be typical.

"Noun" knowledge of procedures. A student may have learned an S-algorithmic procedure so that he can carry it out — he has "verb" knowledge of the procedure — but he may be unable to talk about the procedure, to use the procedure itself as an object of discourse — he "can't use the procedure as a noun." Consider these examples:

1. Prove that the determinant

$$\begin{vmatrix} a & a & d \\ b & b & e \\ c & c & f \end{vmatrix}$$

is zero. Proof (which can be handled without writing anything on paper if you can visualize the relevant procedures clearly in your mind): If I expand by minors (cofactors) of the third column, every 2-by-2 matrix will be of the form

$$\begin{vmatrix} x & x \\ y & y \end{vmatrix}$$

But this will be zero. Now, every term in my sum will contain a factor zero. Therefore the sum will be zero.

2. (From our ninth-grade unit on limits). The notion that $.999\overline{9}$ is somehow a little bit less than 1.000 is wrong. Since we know that every bounded monotonic sequence converges to some number, $0.999\overline{9}$ is the name of a number. By the Law of Trichotomy, that number R satisfies either $R < 1$, $R = 1$, or $R > 1$. By using inequalities we easily show that $R > 1$ is false. By using an approximation, we show that $R < 1$ is false. (Note that $R < 1$ is not approximately false; it is false, period!) Therefore $R = 1$ must be true. (NOT R equals *approximately* one.)

3. (Ninth grade) More generally, limits can give us exact answers, not merely approximate ones, because we use approximations to establish the truth or falsity of certain statements, and these statments can give us exact answers. (For a student to be able to say this requires the ability to check over the proofs in his/her mind, and to talk about these proofs.)

The goal of our ninth grade unit on limits — which involves a detailed use of inequalities, "for any $\varepsilon > 0$ there exists an N such that ...," and so on — is to have students *understand* statements such as 1) and 2) above. This is not the same thing, by any means, as the ability to memorize and repeat these statements.

An analytic break-down of "kinds of thinking" begins to give us a far stronger hold on what we mean by understanding. In this case, "understanding" means that each of the various S-algorithms exist in the student's head as a reasonably well-defined sequence of steps, and that he can compare each se-

quence against the claim of the verbal statement and confirm that the statement correctly describes each proof.

It would be an even more impressive performance for a student to work out this description him or herself, and still more impressive for him/her to volunteer this remark with no suggestion from anyone else — it would presumably be evidence that s/he had spontaneously posed this question in his/her own mind and worked out the answer by thinking through to procedures employed in studying limits.

"Linguistic conventions". Our seventh-grade students can give a correct axiomatic proof that

$$A + (2 \times A) = 3 \times A \ ,$$

but we cannot possibly go through ordinary ninth-grade algebra by proving each identity as a theorem. We need to know, for example, that

$$2x3x = 6x^2$$

$$3 + 2x + 5 - 4x = 8 - 2x$$

$$(A - B)(A - B) = A^2 - 2AB + B^2$$

$$A - (B - C) = A - B + C$$

$$-1(A - B) = B - A$$

$$\frac{A}{B} \times B = A$$

$$(A + B)(C + D) = AC + AD + BC + BD$$

and lots more. These transformations have a rather linguistic character and are known implicitly more than explicitly.

False conventions. It thus becomes a special problem for students to avoid adding to their repertoire many false conventions, such as

$$\frac{a}{b+c} = \frac{a}{b} + \frac{b}{c}$$

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$$

$$(A+B)^2 = A^2 + B^2$$

$$\sin(A+B) = \sin A + \sin B \quad \text{, and so on.}$$

Assimilation paradigms. We place great reliance on precise analogies or metaphors. The idea is that some general pattern exists in the mind, and some new data is assimilated into this metaphor — the data is mapped, in a precise one-to-one way, onto the "assimilation paradigm."

More recently, we have come to explain this phenomenon in terms of Minsky's concept of frames (Davis, 1976).

Imitation of procedures. A sizeable part of learning mathematics consits of observing the demonstration of a procedure -- long division, for example — and then attempting to carry out the same procedure by a kind of follow-the-leader imitation. All students make use of this; the trouble occurs if this imitation is not supplemented by various kinds of inquiry, analysis, and comparison.

Looking for patterns. Consider a class of seventh graders working on the assignment:

1. Graph $y = (x - 2)^2$
2. Graph $y = (x - 5)^2$
3. Graph $y = (x + 1)^2$
4. Graph $y = x^2 - 9$
5. Graph $y - (x - 2)^2 - 9$.

An observer can distinguish (at least roughly) three kinds of student behavior -- tow in this category and one in the next:

1. Solving each problem by making a table, then a graph — with nothing further; no comparing or contrasting, no search for patterns.
2. Solving the problems, but looking for patterns. (David Page's teaching places great emphasis on this step; one can recognize this behavior because the student is suddenly able to graph, say, $y = (x - 7)^2$ immediately, with no recourse to a table.) This is the performance specifically referred to in this category.

But there is at least one further level of performance that some students have exhibited, which must be included in another category:

General cases. Continuing the example from above, there is a third level:

3. Some students pose for themselves the "general case," succeed in writing it with appropriate parameters (in this case $y = (x - a)^2$), and are able to solve it. (In this case: What is the lowest point on the parabola $y = (x - a)^2$, and where does the parabola intersect the x-axis?)

"Fitting things together": Contrasting and comparing. In a recent seventh-grade class, the number $i = \sqrt{-1}$ was introduced by means of the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the Argand system of assigning complex number names to points in the plane was developed. Using the usual FOIL method of multiplying binomials, it was noticed that

$$\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^2 = i \quad , \text{ so that } i^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \quad \frac{i}{\sqrt{2}} \quad .$$

One student, Michael, generalized to the fact that

$$i^{(1/N)}$$

lies on the unit circle, with an angle $(90^{o}/N)$. Immediately, another student, Philip, volunteered the observation that this was then consistent with the expectation that i^{o} should be 1. Philip had evidently checked this new idea against the rule for zero exponents that he had learned months earlier. One wonders how extensively Philip checks each new piece of informaton against possibly-related matters that he has learned earlier. Does such cross-listing behavior at the time of filing new information give Philip an exceptionally powerful retrieval system with an unusual richness of association paths? That would make this a very non-linear phenomenon; much like the rich getting richer, the student with the best developed associaitons could do a better job of cross-referencing each new piece of information s/he encounters, thus increasing even more his or her advantage in the future.

Long strategic planning sequences. The inverse of a 3-by-3 non-singular matrix, like

$$M = \begin{pmatrix} 3 & 4 & 0 \\ 5 & 6 & 3 \\ 2 & 0 & 2 \end{pmatrix},$$

can be found by making use of the observation that

$$\begin{pmatrix} 3 & 4 & 0 \\ 5 & 6 & 3 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix},$$

and similarly for the second and third columns, plus the property that, if T is a transformation induced by a matrix, and if $\vec{u}$ and $\vec{v}$ are vectors in the domain of T, and α and β are any real numbers then

$$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T\vec{u} + \beta T\vec{v} \quad .$$

This seemed to us to have a very nice, very tight strategic planning sequence that told us just how to solve it.

1. If we know the image of the vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

we know the first column of the matrix (and similarly for the other columns).

2. Since we want M^{-1}, that means we want to find $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in the range (or "image space") of M.

3. If we do find $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in the image space of M, as a combination of $\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$,

then we know the inverse image (under M) of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, so we know the first column of M^{-1}.

Our ninth grade students, however, find such strategic planning sequences rather hard to deal with. They easily learn the corresponding procedure for finding M^{-1} as an S-algorithmic sequence of steps, but they do not relate it to this (or any other) planning sequence.

Long execution sequences. Two seventh-grade classes were given this problem:

By graphing two parabolas, find the truth set for the open sentence

$$(x^2 - 1)(x^2 - 6x + 5) < 0 \quad . \qquad (1)$$

Intended solution: let $y_1 = x^2 - 1$ and $y_2 = x^2 - 6x + 5$. Then inequality (1) becomes

$$y_1 \cdot y_2 < 0 \ ,$$

and, for this to be true, both y_1 and y_2 must be different from zero and must have opposite signs. Therefore, graph $y_1 = x^2 - 1$ and $y_2 = x^2 - 6x + 5$ on the same axes and look for those places (i.e., x values) where one curve is above the x-axis, while the other is below the x-axis.

The seventh graders found this very difficult. Every single step along the way seemed simple enough.

Thinking of symbols or thinking of meanings. One of the criticisms that the new math proponents levelled at traditional math was that it dealt with symbols rather than ideas, whereas mathematics properly deals with the entities denoted by the symbols. This distinction has not loomed large in our studies of seventh and ninth graders, which is probably a commentary of some sort. It does happen, however, at least as one checking procedure for judging linguistic rules such as

$$2x3x = 6x^2$$

or

$$(^{-}1)(A - B) = B - A \ ,$$

and so on. Our observations are that most students very rarely use such checking procedures. To outward appearnces, at least, they seem preoccupied with the symbols themselves and rarely think of the entities represented by the symbols.

Critical ability. Marion Walter has pointed out how little practice students get in criticizing someone else's work. Our observations confirm a pronounced student weakness in this area. The textbook we use (Stein & Crabill, 1972) seems excellent in its attempts to give students practice, but they do not find this an easy area.

Stein & Crabill (1972, p. 308), for example, have a character named "Speedy Sam" solve

$$x^2 - 7x + 12 = 2$$

by factoring as

$$(x - 4)(x - 3) = 2$$

so either $x - 4 = 2$ or else $x - 3 = 2$, hence $x = 6$ or $x = 5$. Students are asked to say what Sam did wrong.

All of our students could solve this problem correctly themselves, but not one of them gave a really good analysis of exactly what Same did wrong. This would appear to confirm our growing suspicion that students deal in S-algorithmic procedures and neither relate these procedures to conceptual foundations (including heuristics, strategic planning sequences, etc.), nor think of the procedures themselves as entities (in the "noun" sense) with their own attributes, beginnings, middles, ends, etc.

Distinctions and similarities. One of the main tasks of a mathematics teacher is helping students to realize that certain things, although they look the same, are actually different, whereas other things, although they look entirely different, are actually the same. One of the simplest examples would be the "2" on a ruler vs. the "2" on a clock; these look exactly the same, but are quite unlike. Indeed, since the "12" on a clock is, of course, actually a zero (i.e., $A + 12 = A$), and since, on the clock, $2 \times 6 = 12$, the clock "2" is actually a divisor of zero, whereas the ruler "2", of course, is not. By contrast, the base-ten numeral "2" and the base-two numeral "10" look quite different, but they name precisely the same entity.

The logarithm function, for example, exploits a similarity between additive and multiplicative structures that might easily pass unnoticed.

When to fight and when to switch. If a strategy for attacking a problem is fundamentally wrong, or if enough errors have crept into the work, the attempt must be abandoned at some point. Perservering too long would be a mistake.

By contrast, if the problem really is rather long, one can be on the right track and be 90% of the way toward the goal, and then become faint of heart and quit. Quitting, here, is a mistake. One ought to persevere. (Maybe this should be named after the "Sail on! And on!" poem about Christopher Columbus.)

* * *

Obviously, there are redundancies in the preceding list of information-processing procedures. As suggested earlier, this analytic effort is more nearly parallel to Lavoisier's position at the historical beginning of classifying chemical substances than to Mendeleyev's much later successful effort at organizing elements in the periodic table. We haven't gotten very far, but we feel it is imperative that a start be made. It is simply unproductive to talk about mathematics in such large (and inhomogeneous) chunks as "ninth-grade algebra" and tenth-grade synthetic Euclidean geometry, with no concern for the great diversity of information-processing tasks that in fact constitute mathematics. One difficulty of the new math era was that people were unable to articulate what it was they wanted students to learn.

Some Examples

With the preceding list of information-processing tasks available for reference, it may be worthwhile to consider a few examples.

Example 1

Consider an earlier example of a class of seventh graders graphing parabolas. The students had graphed $y = x^2$ carefully on heavy paper and cut it

out, so that they could use it as a stencil to draw accurate parabolas. All of the parabolas in this discussion had exactly this size and shape, so that the exercise in graphing parabolas — in this particular lesson — focused on two questons: Where is the lowest point of the parabola, and where (if anywhere) does it cross the x-axis?

As mentioned previously, three levels of student performance could be distinguished. Some students solved each problem separately — one at a time — and did not compare them or look for patterns. Thus they graphed

$$y = (x - 2)^2 \quad ,$$

then graphed

$$y = (x - 5)^2 \quad ,$$

in each case by making tables of values. When they then encountered

$$y = (x - 4)^2$$

this appeared to them as an entirely new problem. They made a table, plotted points, and made a graph. (It was partly to help them to see the similarities in these problems that the stencil was made.)

Other students saw the pattern almost immediately. Some of them had conjectured the pattern correctly from a single instance! This sequence is perhaps the most interesting. It led up to the question: What is the lowest point of the parabola

$$y = (x - a)^2 - N^2 \quad ,$$

and where does it cross the x-axis? One can solve this either by generalizing from instances, or by an analytic solution. From examples, one saw (if one noticed!) that the replacement of x by x - a shifted the curve a to the right, while the $-N^2$ term lowered it by an amount $-N^2$ (or N^2, if you want to consider distances as positive). The problem could be solved in this way.

The physicist David Bohm made an important analysis of scientific thought in which he distinguishes "insight," on the one hand, from analytic, algorithmic,

computational hypothesis testing. Bohm pointed out that traditional analyses of science have placed great emphasis on hypothesis formulating and hypothesis testing — too much emphasis, he argued. Insights are rare and precious, but hypotheses are cheap, he remarked. As one example, Bohm considered Newton's studies of gravity. The Greek theory of heavenly substances moving in exalted circles, and earthly materials seeking appropriately low levels, gave no reason to ask, "Why doesn't the moon fall?" One might as well ask, "Why isn't the king a peasant?" When Newton had the insight that the moon was like an apple (or whatever earthly metaphor he actually used), the question, "Why doesn't the moon fall?" became inevitable. When he answered, "Actually, it does fall!", he had completed the main insight. The task of adjusting a precise hypothesis to explain the elliptic orbit was now merely a technical exercise.

It is worth asking whether Bohm's insight in the parabola case might not be the same as our generalizing from instances, and/or our "assimilation paradigm." It may even be profitable to analyze it in terms of Minsky's frames (Davis, 1976). There is certainly a metaphor or gestalt quality about it. I can graph

$$y = (x - 7)^2$$

almost immediately, almost without thinking, because I recognize it.

The analytic approach is quite different, I can say

$$y = (x - a)^2 - N^2$$

means that, since $-N^2$ is constant, y is a minimum when $(x - a)^2$ is a minimum. But since $(x - a)^2$ is a square, its minimum is zero;

$$(x - a)^2 = 0$$

implies

$$(x - a)(x - a) = 0 \quad ,$$

which, by the "zero product property" implies

$$x - a = 0 \quad .$$

So the lowest point is $x - a$, $y = -N^2$, or $(a, -N^2)$.

Notice, though, that even this analytic approach seems to require the use of insights at certain critical junctions. In studying examples like this, it is easy to get the notion that metaphors, assimilation paradigms, or frames are a fundamental part of the vast majority of calculations. You can't calculate how fast the moon is falling if you never realize that it *is* falling; the calculational sequence must be initiated by a metaphor or insight.

Example 2

Ninth-grade students had at least modest success in writing an inverse matrix for

$$M = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

by guessing (essentially) that in the x-dimension things are tripled, whereas in the other two dimensions there is a rotation through 90° counter-clockwise. Hence, the inverse matrix should be

$$M^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

This may be another instance of insights instead of calculations.

Example 3

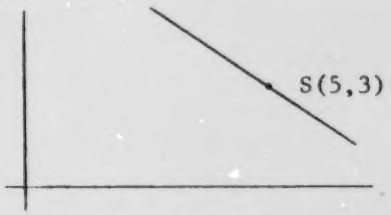
One tenth-grade geometry textbook has eight consecutive problems of the identical type: "Write a parametric vector equation for the line through the point S parallel to the vector V." These eight identical problems hardly seem to encourage much thought, but watching Ann, a tenth-grade student, do her homework is very revealing.

In the first place, Anne is in a hurry; she has worked out a schedule and has allocated 20 minutes to this mathematics homework.

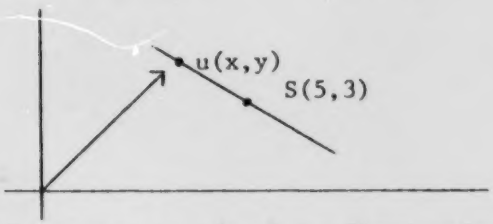
In the second place, the observer had the very distinct feeling that Ann did not want to think about this mathematics any more than the irreducible minimum amount. Dismiss this as impressionistic if you insist; the events of the next few minutes speak for themselves.

1. One could do the homework problems on a conceptual basis, for example, by saying:

   Here, say, is the required line:



   I want a vector to a general or moving or variable point on the line, call it u(x,y):



   I can get to u(x,y) by first going out to the line (which I can do by going to S, a point I know lies on the line), $\vec{u}(x,y) = \vec{S} + \ldots$ and then moving along the line by an "adjustable" multiple of $\vec{v}$

$$\vec{u}(x,y) = \vec{S} + r\vec{V} \quad ,$$

where r is a real-valued parameter.

The instructor had urged this conceptual approach. Ann rejected this and sought instead to imitate the formula shown in the book. "Imitate" is the word -- she made no effort to analyze or interpret it. She merely said, "Let's see, $\vec{S}$ is (5,3) and $\vec{V}$ is (3,-1),..."

2. In a later problem,there were more complications -- finding the three altitudes of a given triangle. The teacher had suggested drawing a careful picture on graph paper (which immediately gave a good estimate of the answers). Ann chose not to make a picture because "that would take too long." Instead, she looked for a formula in the book to which she could apply her imitation process. Finding none, she was finally led to making a picture. But instead of a careful picture on graph paper, she made a careless picture with so many distortions that it confused her.

When she asked for help, the teacher tried to analyze the problem with her. The teacher said Ann responded to this approach by saying impatiently, "I don't want a speech, I just want to know how to do the problem!" When the teacher finally found one altitude, Ann quickly imitated that solution to find the two others.

3. The book dealt with lines in a variety of ways, specifying them either in two dimensions or in three, either through a point S and parallel to the vector V, or through a point S and perpendicular to the vector n (in 2-D) or to the plane $\mathcal{P}$ (in 3-D), or through the point S and parallel to the line $\mathcal{L}$, or through the points S and R, and so on. Answers were sometimes asked for as a parametric vector equation, sometimes as parametric Cartesian equations, sometimes as an equation in the

slope-intercept form, etc. The teacher suggested that the students try to organize all those alternatives sytematically and compare them. Ann refused to do this, saying, "There isn't time!"

By not comparing and contrasting the various situations by refusing to think about a conceptual approach, by making the fewest possible diagrams (and then only sloppy ones), by refusing to try to analyze the formulas — indeed, by limiting herself to imitating a formula in the book — Ann was making it very unlikely that she would ever learn or understand the content of the chapter on lines. But this lack of understanding must surely be cumulative. After a while, Ann will have no choice. She will have closed out all possibilities of "understanding," and she will be compelled to rely (as, in fact, she already does) on her method of imitating illustrative examples or formulas in the text. Ann's behavior is *so* imitative that she can only look at a line or step in the book and try to imitate it; she is not even memorizing an S-algorithmic procedure. In short, she cannot solve such problems with her book closed.

Example 4

Some eleventh-grade students complained recently about their tests. The tests, they said, were much too different from the homework and it wasn't fair.

As a matter of actual fact, every test item was taken from the homework assignments — one problem from each of five different assignments. But that was the secret! Within a single homework assignment, all problems were of a single type. Students were not getting "diagnositc" practice in looking at assorted problems and deciding how to solve them. We have responded to this by giving some homework assignments that consist of a variety of entirely different kinds of problems. The super-procedures that select and sequence sub-

procedures are, as our student interview data shows, the most common source of failure or errors;no wonder, if students try to avoid asking, "Why am I solving this problem in this particular way? How would I know to try this approach?"

Example 5

We allow students to retake exams (using different forms of the exam, of course) until their grade is as high as they want it to be. There seems to be a difference, however, between what is learned by those who usually make 100 on the first version and those students who need several retakes. It seems that the multiple-retake students rely more on learning, from the exams, just what is expected and memorizing that. By contrast, the "no retake" students seem to aim for a general understanding of the material. One suspects that the "no retake" students get more deeply involved in the material, give more of themselves, and will remember the material far longer.

Comment on Advanced Content

The content in these illustrations may seem unusual. In one respect, at least, it is; most (but not all) examples are from our laboratory school, which has a special math curriculum for gifted students. Seventh-grade students begin with careful proofs of algebraic theorems, then move on to Stein and Crabill's(1972) ninth-grade algebra book (which includes differentiation of polynomials, slope, and maximum-minimum problems). To this is added some work with hand-held calculators, study of formal logic, and work with matrices, vectors, and complex numbers.

There is no eighth grade. After grade seven, the students move directly into grade nine. Grade nine continues the themes of grade seven, adding a careful study of limits of infinite sequences, an extensive study of systems of equations, some trigonometry, the solution of differential equations by graphical and numerical methods, and further study of linear transformations

and complex numbers.

We have been experimenting with versions of this curriculum for over 10 years, and we are convinced that the usual pace of the curriculum in U.S. schools is far too slow, which probably does not make life easier for students in the long run. The very slow pace is probably related to an attempt to reduce all of mathematics to S-algorithms and to have students learn it that way. We doubt that such an attempt can ever be made to succeed. Where it appears to succeed, it is probably because the best students, in the sense of Beberman, Page, Groen, and Resnick, have been thinking about mathematics in a way quite different from that which was being taught.

Pathology

We have become interested in observing many kinds of obstacles to learning; quite a few are easily observed. Some students leave books in their lockers at school and, therefore, cannot do homework at home. Similar to this is the behavior of simply not sitting down to try to do the work. No matter how bright or able a student is, s/he will not know what's in a book that's never opened.

Because of the ease of observation, we have also looked at the students practicing the piano. The following practice session is typical for Ann, the same student whose studying of geometry we recounted earlier. She is studying a classical sonatina by Diabelli. In fact, she might better be described as "attempting to play it" — she is not sufficiently attentive to be described as "studying" it. She opens the book and immediately begins to play — not, it turns out, having even noted correctly the key signature. In the third measure, there is an error in the left hand, which she does not appear to notice. She plays with an irregular tempo, suggestive of no recognizable rhythm. By the third line, she has become aware that it sounds funny, and when she evidently and unmistakably misses a chord, she stops. She does not analyze the chord or

any of her errors. Instead, she jumps back to the beginning and once again starts to play, with no greater care than before. The third time she misses the chord in the third line, she abandons the Diabelli altogether and starts to play the beginning of Joplin's Maple Leaf Rag, a piece she had played quite well a year earlier. But now she is careless, and the Jobplin is hurried, uneven, and unsatisfactory.

There are many important and necessary things Ann fails to do. She never analyzes her mistakes, and she never pays special attention to the more difficult passages. Indeed, in three months of practicing one Diabelli sonatina, she never once played the most difficult two measures; she always ran into trouble earlier in the piece and, therefore, always went back and started at the beginning. The difficult part could have been learned only by some kind of miracle, which in fact did not occur. Nor, of course, did unnoticed or unanalyzed errors miraculously go away. Yet when Ann made up her mind to learn a piece (about once a year), she could learn it very quickly — in a few days — and play it reasonably well. In this way she had learned Maple Leaf Rag, Mozart's C major sonata (K. 545), Beethoven's Für Elise, and Kabalevsky's Op. 27 No. 7.

It is certainly suggestive that Ann's method of avoiding the essential aspects of practicing the piano is similar to her method of avoiding the essential aspects of learning geometry. From our own observations of this type, we have seen one mystery disappear to be replaced by two others. Why some students learn so much faster and better than others is no longer mysterious to us; some students go about practicing and studying in ways that are effective, and some do not. The two new mysteries are *why* some students have effective study methods while others do not, and what can be done to help those students who do not yet have effective ways to learn.

This same phenomenon has been discussed in other contexts. In his remark-

able book, Zen and the Art of Motorcycle Maintenance, Pirsig (1974) describes a garage where he took his motorcycle for repairs. The mechanics' original diagnosis was hasty — too hasty, as it turned out. They immediately diagnosed the trouble as tappets. Some weeks and $140 later, the cycle was in worse shape than it had been. Their last attempt to fix it ended when Pirsig came to pick it up and found it covered with grease. He wiped it off. It wouldn't start. He checked the electrical system, found some wires that had been left unconnected, connected them, and started it. Now the tappets really *were* a problem — they hadn't been adjusted properly. Pirsig pointed this out to one of the mechanics, who came over with an inappropriate tool — an adjustable open-end wrench — used it improperly (with a wrong setting), and rounded the tappet covers. Then the mechanic got a hammer and cold chisel and started to pound them loose — and pounded a hole through the aluminum cover. Then, the hammer missed the chisel altogether and broke off part of two cooling fins. Prisig took the cycle "as is" and went home and repaired it himself.

Later on, he says,

> I tried to think back into that shop, that nightmare place, to try to remember anything that could have been the cause.
>
> The radio was a clue. You can't really think hard about what your're doing and listen to the radio at the same time. Maybe they (the mechanics in the garage) didn't see their job as having anything to do with hard thought, just wrench twiddling. If you can twiddle wrenches while listening to the radio that's more enjoyable.
>
> Their speed was another clue. They were really slopping things around in a hurry and not looking where they slopped them ....
>
> The biggest clue seemed to be their expressions. They were hard to explain. Good-natured, friendly, easygoing — and uninvolved. They were like spectators ....
>
> There was no identification with the job. No saying, "I am a mechanic." At 5 p.m. or whenever their eight hours were in, you knew they would cut it off and not have another thought about their work. They were already trying not to have any thoughts about their work *on* the job. (pp. 25-6).

So the unequal results are no mystery at all. But why are the approaches, the actual activities so different? That is indeed a mystery.

Kinds of Students

Without pursuing here the question of different kinds of students, we report a remark from one experienced and successful teacher: "I make a distinction between the students who are trying to do the homework, and the students who are trying to learn mathematics."

This remark is readily interpretable in terms of the categories of behavior listed earlier.

Intent

We find ourselves driven to the idea of "intent." By intent we mean something different from the usual notion of motivation; intent is much more sharply focussed.

Consider Ann practicing the piano. Was she really trying to practice? Was she trying to go over the piece until she knew it? Was she trying to identify difficulties and concentrate her efforts efficiently on overcoming the difficulties?

Notice that intent is shaped by many things, not least by the learner's belief as to what is possible and how it is to be achieved. Notice also that, for this reason, intent is not simple to influence, either by student or by teacher.

Pirsig (1974) seems to have come to much the same conslusion. After considering the kind of manual that could be written to aid in motorcycle repair, he concludes that everyone seems to be missing the real point:

> It occurred to me there *is* no manual that deals with the *real* business of motorcycle maintenance, the most important aspect of all. Caring about what you are doing is considered either unimportant or taken for granted. (p. 27).

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A Reaction by Robert G. Bridgham*

My first reaction when I encountered the paper that Robert Davis prepared on "Curricular Implications for Research on Teaching Mathematics" was "Where is the curriculum?" My second reaction was "Where is the teacher?" My third reaction was that both are there, but present as background to what is in the foreground of Davis' interest--the student's response to opportunities to do mathematics. Davis' verbal presentation was less characterized by an almost exclusive attention to students and student responses, than his written paper, but an emphasis on student responses to mathematics market it, too.

As I pondered the paper, the analogy of a craftsman who is confident of his design ability and technique, but conscious of some refractory qualities in his materials came to mind. He focuses on what is happening to the handiwork as he manipulates it, and tries to figure out what in the material is producing the results that he sees. In the image that he attends to, there may be his fingertips at the periphery, but at the center is the work itself as it develops. When his attention is concentrated in this,way, the craftsman's description of what is happening will include very sparse and partial accounts of his own activity--only as much as is needed to identify and make sense of what is happening to the material being worked. This is precisely what Davis has done.

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I suspect that Davis has a well-warranted confidence in his command of the subject matter and of the possibilities that mathematics offers for educating children in the middle grades. I suspect, too, that Davis can be confident in his control of the instructional "moves" useful in the teaching of mathematics. All of that fades into the background in his account, however, and what we see in the foreground is the child, the student, coming to the mathematics, trying to make sense of it. Davis gives us a catalog of the different ways in which children respond to mathematics. What he finds problematic is the student's response to mathematics and it is this that he therefore makes central. The description is of the student being taught; Davis himself as the teacher, is harder to see in his description.

In this paper, Davis used a metaphor that I was very taken by. It connected with something in my own background. He suggested that in the work that he has reported here, he and his colleagues are doing something like Lavoisier's classification of chemicals into elements and compounds. What Davis has offered us is a description of elements--the elements of student response to mathematics instruction.

The chemical elements can be thought of as falling into two major categories, electron donors and electron receivers. It seems to me, by similar token, that a more adequate description of what goes on in the classroom in mathematics instruction will have differentiations for the teacher, and for teacher action, similar to what Davis has presented for the students. When you look back at the examples that Davis presents, the student response is coupled with a task that the teacher has set. It

may be productive to look at the setting of tasks if we want to differentiate critical elements of teacher action.

What are the kinds of tasks that teachers set when they ask children to do mathematics? I started to make a list suggested by the examples that Davis had given. Sometimes the task is to find something, such as the solution to the equation $\frac{3}{x} = \frac{6}{3x - 5}$. Sometimes the task is to prove something; for example, "Prove that A + A = 2 X A." Sometimes the task is to represent something; for example, "Graph $y = (x-2)^2$." Sometimes the task is to check or criticize something, such as a propcsed solution of $x^2 - 7x + 12 = 2$, that through the factoring (x-4) (x-3) = 2 leads to a claim that x=6 or x=5.

I'm not sure that the characterizations of tasks that I've offered fit the domain well. That I would want mathematicians to consider. I am certain that the list I've produced is incomplete. There is a value, though, to thinking about descriptions of the kinds of tasks that are set for students along with thought about the responses of students to these tasks. You then begin to worry about the kinds of questions that chemists have entertained. What combinations of elements produce stable compounds? (Which kinds of tasks in mathematics will most persistently elicit certain kinds of student responses? Which kinds of tasks are almost never coupled with certain kinds of student responses?) What task-response compounds are comparatively unstable, coming apart with even small perturbations?

If we push the chemical metaphor a bit further, we note that chemists also try to understand the conditions under which reactions

occur. They talk about ideas like "activation energy," about factors that influence the kinetics of reaction, about how quickly reactions will go. If you look closely at the examples that Davis has given, you'll see that his teachers are manipulating the conditions in which tasks will be completed.

One of the things that teachers do is probe. When Davis talks about developing ad hoc heuristics, he describes a series of probes, each indicating what the student should be thinking about at the time. The entire series of probes is a model of the heuristic process. Teachers also offer various hints. They offer worked examples; they issue warnings as to "how you might go wrong," they exhort to action or to perseverance ("sail on").

One of the critical things that teachers do is arrange mixtures and sequences of tasks. Robert Davis describes asking students to graph six successive equations for parabolas with the hope that by the time they reached the sixth grade, they would begin to generalize and perhaps be ready to state for themselves a general case. Davis also mentions the importance of mixing different kinds of tasks so that students don't simply repetitively apply the same kind of S-algorithms, but must choose which ones are appropriate to each particular task.

Teachers are continually acting in a variety of ways to modify the "task environment" of their students. We need to work out effective characterizations of these "condition changing" acts of the teacher in addition to characterizations of the teacher's task setting and the students' responses.

If we manage to work out the needed characterizations and differentiations, what does this yield for research on teaching mathematics? To the extent that the differentiations are sensible, they will provide a language for describing how teachers concentrate their curricular thought. What kinds of tasks do teacher entertain as possibilities, what kinds of instructional prompting do they think of using, what kinds of student response do they hope to stimulate? When teachers have a limited understanding of mathematics and limited practice in teaching the variety of possible mathematical "moves" (I suspect this is true of many elementary school teachers and that they are conscious of it), how do they characterize their limitations and the consequences of these limitations? What do teachers imagine to be the possibilities for themselves, in teaching mathematics, in terms of their own sense of who they are and what they can do?

I see a curriculum as a plan for providing educational experiences, accompanied by detailed supporting arguments, and sometimes associated with the materials designed to forward the plan. It is clear from this definition that different people will formulate different curricula. One aspect of research on teaching mathematics is research on how, when, and in what terms the classroom teacher formulates curriculum for mathematics. Moreover, how does the curriculum of the textbook author or "commercial" curriculum designer affect the curriculum of the teacher? This is a problem stated in terms such as "adoption" and "dissemination" when those who prepare materials are thought to be more trustworthy than classroom teachers.

When we credit the teacher with significant and valuable knowledge relevant to mathematics teaching, as I think we must, the terms of the problem change, but the problem remains.

We have not yet found secure ways of arranging communication between the designers of curriculum materials and classroom teachers which acknowledges the strengths of each, provides for effective collaboration in materials design, and which doesn't require that all the adapting and conciliating be done by one of the partners. One of the reasons this problem remains is that we lack good descriptions of teachers' curricular thought. What is it that teachers think about? How do they decide what to do, how to do it, what to hope for? What do they take into account?

Effective research in mathematics teaching will tell us something about teachers' curricular thought and in doing so will form implications for the development of curricular materials. The implications for curricula from research on teaching are an appropriate complement to implications for research on teaching from curricula. Between mathematics curricula and research on mathematics teaching, the currents should run in both directions.

A Reaction by Joseph J. Schwab*

Rather than comment about what Davis has written, I am going to talk about him as a superlative example of a teacher who has become acutely aware of the data that classrooms and students provide and has tried to interpret them. This is extremely pertinent to the IRT's interest in information processing by teachers with respect to a single point, representing to me the most difficult task faced by investigators of information processing. Davis states that "Verbal knowledge -- information coded in a natural language statement -- is presumably a quite different type of knowledge"(p. 48), and it is very different from S-algorithmic procedure or mathematics generally. Then he gives a number of examples. Looking at these examples, I discover that they are not examples of a great difference between mathematics as a language (a concept which is, by the way, questionable) and a natural language statement, but are simply, in two cases, examples of very imprecise use of natural language. In one example, Jennifer, a third grade student, claimed 8, 80, and 800 were all the same because 0 means nothing; "adding 0 doesn't change anything" (p. 48). Now, if the teacher had made such statements, the teacher was exceedingly stupid and was at fault, not the natural language. If the teacher

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had said that arithmetic addition of 0 as a number adds nothing to the array of numbers to which it is added, but that adding 0 as a number doesn't mean putting it on the page beside another number, obviously the student would not have made the same error.

A second example of the same kind is the case in which several seventh-grade students had graphed $2X^2 - X^2$ and got the answer 2, reading the problem as $2X^2$ take away X^2 leaves 2. Surely, in natural language the teacher might have said, "When a term in any equation has no numerical or quasi-numerical prefix, it is understood to be 1." Then, of course, the student would have read $2X^2 - 1X^2$ and the answer might have been $1X^2$ left over.

Now, the important thing to me is not the imprecise use of language by teachers, but the deep algorithm in Davis that led him to interpret these cases as cases of the fundamental inappropriateness of natural language to the teaching of math. I can make guesses; the most obvious thing is that Davis is a mathematician. Yet, the real problem posed has nothing to do with Davis and has nothing to do with the slips, because we all make them. The real point has to do with how investigators of information processing probe to discover the hidden algorithms by which meanings are imposed on the data which are used.

In another case cited, Davis mentions the preference of children for the S-algorithms approach. Davis suggests that, "Perhaps this is a residue of a bad teaching approach in earlier grades that has produced an addiction to S-algorithmic methods?" (p. 44). Now there is something a bit general about this example. Consequently, my interpretation cannot be precise. I interpret it to read, "Is this a residue of the bad teaching of mathematics?" My defense for that interpretation is, "Did the bad

teaching of mathematics cause the problems?" In this interpretation, we see again the operation beneath algorithm. Davis's statement does not take into account the possibility that all the teachings, including the teaching of the so-called natural arts, have been atomized and technicalized. Nor does it consider that students have been so firmly taught from the beginning, in all classes, that conformity and obediance are vastly preferred to experiment and risk-taking (both of which are always involved in the conceptual matter). There is nothing left for them to fall back on in order not to prefer simple algorithmic methods unless they have a highly supportive home.

A Reaction by Edward Smith*

The first point made in Davis' paper was that in teaching there is an interaction between two distinct things -- the kind of thing that can be learned and the way in which it is taught. This distinction is also reflected in the two taxonomies that Davis proposed, one dealing with kinds of knowledge or skills to be required, and the second one with kinds of learning experiences.

The term "curriculum" has always been difficult to define, and I think this difficulty is evident in these deliberations. Bob Bridgham's definition of curriculum includes what is being taught as well as how the teacher teaches it. Davis would certainly recognize that these are very intimately intertwined in practice, but I think he is proposing that researchers differentiate between the two in analysis. This is an important distinction, one which I think should be an issue for further consideration.

A second point made in the paper is the need for differentiation of subject matter. Rather than looking at teaching, or even the teaching of math, as a generic activity, Davis proposes a differentiation of what is being learned and how those particular entities may be taught by teachers and learned by the individual students. This would result in acquiring knowledge at the intersection of variables such as student characteristics, subject matter, learning activities, and so on.

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Davis proposed this differentiation very clearly in an earlier paper (1976) on the idea of education as applied science. There, he contrasted the notion of scientific inquiry as a rather narrow, theoretical undertaking (such as might occur in classical physics) with another kind of applied science (such as engineering, for example). Whereas physics is characterized by a small number of elegant idealized laws and principles, the key characteristic of engineering or similar applied sciences is the amount of detailed information that has to be catalogued and made available to the practitioner. The engineer is faced with somewhat different tasks than the physicist. Davis characterized the engineer's task as trying to make something that works. Likewise, the teacher, the curriculum developer, and the teacher educator are all attempting to design something that works in the classroom with kids. Preoccupation with the theoretical, "physics-like" undertaking tends to give those of us involved with educational research a somewhat well-deserved label as being "irrelevant," "ivory tower," "too abstract," and so on.

My next point deals with the major content of the paper -- the list of things that I interpret as those which might be learned through the teaching of mathematics. The categories that are identified are broad operational categories. These categories represent <u>what</u> you do, but what you do these things <u>on</u> or <u>to</u> does not seem to be catalogued anywhere. I think that there is a fundamental distinction between a representation of something and an operation that acts upon it. It would be helpful to have here some categories of structural knowledge to which heuristics, strategies, and other operational categories would apply.

Terms like "variables," "equation,""operator," "place value," and "real number" point to structural concepts. Greeno (1976) demonstrates that representations of quantity can be operated on using the same general algorithm.

The notion of quantity can simply be represented symbolically with a number, or it can be represented in terms of a set of objects. A third representation could involve a notion of area covered, with ½, for example, being something whose area is half-covered. Thus, we are speaking not about operation but about that which is being operated on.

Several constructs have been developed in contemporary cognitive psychology that may be of use in thinking about structural knowledge. One that Davis mentioned was the notion of a frame -- a structure of knowledge that has fixed and variable aspects or "slots." One can apply a given frame by inserting new items in the slots. One of the most exciting potentials for the frame construct is the way analogy might function. A person with a frame may be able to recognize an analogy by putting new specifics into that frame. Nowhere in the paper, however, were there any categories that might represent something like frames or semantic networks. These frames are not identical with the heuristics or algorithms, but rather represent something more structural in nature. The notion of a semantic or relational network is also applicable here. There may be certain types of content that can be more adequately represented in terms of networks, and at least some of the methodology associated with networks may be useful.

My last point concerns students caring about or wanting to understand mathematics. Davis characterized many students as being highly resistant to anything beyond the learning of simple algorithms. This is extremely important when we look at teacher preparation. Some of the students in our science courses for elementary education majors demand this same kind of algorithmic answer. If we don't break through that kind of uncaring and disinterest in true understanding with these preservice teachers

before they go out and influence the values of kids in the schools, we are producing a product that is working exactly against us. Two things came to mind when I was thinking about this problem. One is David Ausubel's notion of a "set to learn meaningfully." Although not given much attention in the research, it is a key element in meaningful verbal learning. Perhaps this aspect of Ausubel's formulation should be examined more closely. Another was Pirsig's (Zen and the Art of Motorcycle Maintenance) notion that caring and seeking to understand are necessary conditions for coping with technology. This can serve at least as a metaphor for our work in this area.

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Discussion*

Davis: People outside the field of mathematics seem to think that S-algorithms are mathematics. In solving a problem, you find out what to do, you do it, and it works. In reality, however, you often have in your head lots of different things you can do -- you have a lot of behaviors you can access. Very often a student is able to do a math problem correctly, but he doesn't access the correct information with which to do it. He may access information on how to do a different problem only to find that it doesn't work. You can't prove on a test that a kid is unable to do a problem; what you can prove is that the information he accessed wasn't right compared with that which he could have accessed.

Bell: Kids don't have a variety of information to choose from in solving arithmetic problems on the elementary level. Elementary teachers seem to do an extremely good job of teaching not so much the wrong things but the right things insufficiently -- such as teaching multiplication as repeated addition. A few days down the road it turns out that multiplication isn't repeated addition anymore. Another example is subtraction; subtraction is taught as "take away," nothing else.

Davis: I think Max just said a devastatingly important thing there. It seems a fair interpretation of the data that we are getting that kids learn some of these things with too much priority. They learn them as absolute truth. The standard error kids make when asked what 7 X 7 is, is to answer "14." When kids learn 7 + 7, they learn it so well

*This discussion, based on audiotapes of the conference, has been paraphrased and condensed to include only key points.

they tell you the sum when you ask for the product. There is a visual similarity, unfortunately. The signs used aren't very distinct. This is a very common error -- even kids who have 140 IQ's say 7 X 7 is 14. Another common error is to say that 3^2 is 6. It seems to be a bad mistake that kids are getting something so clearly that whenever they see it, they always respond in a certain way. Maybe it's nice to introduce multiplication as repeated addition, but you have to keep re-examining such a premise in light of changing ideas. Yet,this is hardly a prominent theme in the math curriculum -- that ideas can evolve. A very important notion in any subject is to say, "okay, let's re-interpret -- go a different way." This constitutes one of my objections to the teaching of sets in the primary grades. Sets arrived historically later in mathematics when man wanted to re-analyze and re-interpret what he was counting in order to get sort of a 1-to-1 correspondence. Initially, however, he counted. We shouldn't ever impose a late version of something too early because it kills the process of changing ideas.

Fey: The magnitude of the problem is fantastic because you have instances of the same kind of confusions in the high school. A couple of years ago there was a continual harangue among high school math teachers about the definition of an angle. They were very uncomfortable about it.

Davis: But the definition of an angle is surely one of the big rules of mathematics. You know the wisecrack about New York City -- when they finish building it, it will be a wonderful place. Mathematics is like that, and a person's understanding of math is like that. It's perpetually changing. Anytime you find you are not changing, re-interpreting, and establishing new connections -- that's when something is wrong. Ideas

must evolve. Yet, the curriculum tries to fixate on the "one true meaning" of addition, an angle, etc. Since the curriculum does it, that's what teachers think they should do. I think we all understand the pressure to do it. If I settle on teaching one thing successfully -- I can teach you one thing; I can teach you what addition is. But that is very counter-productive down the road.

If we're not careful, our study of teaching could come up with the idea that since a teacher sets out to teach multiplication as repeated addition, and by God, he/she does, he/she must be a successful teacher. Yet, a teacher can do something so well that he/she drives out the possibility of kids speculating about alternatives.

Schwab: Someone said that a teacher tries to make out of a kid a machine that works well in the classroom. Well, I don't think that's true, and it ought not to be. The product which the teacher is trying to contribute is not something which works in the classroom, but a child who works well when he is outside the classroom.

Davis: The teacher gets rewarded for making multiplication simple, having parents and teachers all believe it is simple by showing that the kids have learned it. Nobody comes along years later when multiplication needs to be re-interpreted to say that those kids are unable to re-interpret it. Teachers are too effective in teaching a first idea too firmly.

Esty: Even harder to deal with are elementary education majors in the university. These people are harder to break through to than seventh graders. Why? How does this resistance to working at something in a meaningful way develop? I've had similar experiences dealing with seventh graders who have been very successful in their elementary careers -- they are very resistant to looking at things differently

because they already have methods that work.

Schwab: Perhaps the most important factors that contribute to such an attitude are too much reward for obedient following of instructions and too much punishment for experimenting.

Davis: Yes, a lot of it does come down to too much reward for conformity and too much punishment for experiment. And this has to do with the kinds of punishment and rewards kids are getting at home, also. Studying the home environment along with studying teaching would make our results much more intelligible on this point. Looking just at a kid's behavior in school, it is difficult to make any meaningful conclusions as to the causes for his/her behavior. The details provided by knowledge of the home environment may show why a kid acts the way he/she does. Sometimes the teacher is trying one thing and the parent is trying the opposite.

Porter: The point about the home environment's influence on behavior is well taken, but it seems that in the domain of mathematics, the school is not really being challenged by the reward and punishment setup of the home. Yet, the school does reinforce a reward for conformity and punishment for experimental idea by its own use of standardized and teacher-made tests. The tests we examined tested for specific content very algorithmically.

Davis: I'm sure most of you are familiar with MAA tests. They are an entirely different type of mathematics test with an emphasis on novel problems. The aim of the people who make up the test is to have every problem not remotely resemble anything the student has ever seen. The question is whether he/she is good enough at things like heuristics to fight his/her way through a totally novel problem. The MAA program is

looking for mathematic geniuses, trying to find the eight best mathematics students in the United States to send to international competition. Imagine the impact on high schools if more of this type of test were used. Introducing these kinds of problems would make it clear that there is more that's being taught in math than algorithms. Problem analysis is part of the game.

Walter: We're not giving the teachers any leeway to consider what to teach. I think all the problems we've talked about could be partially remedied by doing so.

LeBlanc: There is a tremendous need for some leadership in terms of what might be an appropriate curriculum structure for elementary school children. I am also concerned about flexibility versus the S-algorithm idea. Children do feel comfortable doing certain tasks in which there is some closure. Learning subtraction as "take away" initially may or may not be important. If one visits a first-grade classroom and watches what happens when a teacher is trying to teach subtraction, one will find that it is very difficult for the teacher not to say "here is how you do it" so a certain segment of the children can have some success. The difficulty we have is to not categorically deny S-algorithmic teaching methods, but rather to try to build in flexibility. We need to encourage the learner to move away from S-algorithms and from restricted information to evolving ideas. Yet, I don't see that it is realistic for the teacher to teach S-algorithms as well as to maintain the flexibility for idea connections we say are important.

Davis: That's true, for if you say, "I am not going to teach multiplication

as repeated addition," you may end up not teaching anything at all. There needs to be a balance between the two-- algorithms as part of an evolving mathematics knowledge. The problem is really that people don't have this notion of things evolving. The number 2 will change its meaning as you go through mathematics. It's defined one way first and later on it is defined differently. The best writing I have seen on this subject has been done by anthropologists about what they sometimes call "central definition." I'm sure most things are defined centrally; that is, kids know what a tree is "sort of," but they don't really know the dividing line between a tree and a bush. They know what a cat is "sort of," but they don't really know what the dividing line is between what is a cat and what isn't a cat. Definitions are vague initially. As long as something has lots of the right attributes you consider it part of that class. This idea has a great deal of applicability to mathematics teaching. Often a teacher doesn't need a careful definition of something in the early stages of teaching mathematics -- a central one will do. In this way,the teacher can avoid defining concepts so rigidly that they become imprinted as absolute truth on a child's mind. Thus, the teacher and student can use the concept at the present level, leaving in enough flexibility for a change in definition at the higher one.

RESEARCH ON TEACHING MATHEMATICS TO THE ELEMENTARY PUPIL

Jeremy Kilpatrick

Abstract

Kilpatrick's introductory remarks make a case for more descriptive research in mathematics education. He states that what is needed is a synthesis of the various existing schemes for analyzing classroom behavior plus the generation of other schemes which would be specific to mathematics education for choosing individual differences in pupils as an area for study. He elaborates this by setting forth the aptitude-treatment-interaction model as a framework within which to investigate the problem of bringing individual differences into the research. Also discussed are various mathematical abilities that might allow a clearer look at individual differences, keeping within the information-processing approach. The abilities outlined are from Krutetskii's work. The paper then deals with the teacher as a decision maker. Research should focus on both the decisions teachers make and on the set of alternatives and criteria used in arriving at the decision, according to Kilpatrick. In closing, Kilpatrick stresses the importance of looking at the "master teacher" to begin formulating a cognitive map of what the effective teachers do in teaching.

About the Author

Jeremy Kilpatrick is a professor of Mathematics Education at the University of Georgia. He received his Ph.D. at Stanford University in 1967 and has since worked to broaden the scope of research in mathematics education. Of note is his work with Isaac Wirszup in editing a translation of Krutetskii's The Psychology of Mathematical Abilities in School Children and other Soviet studies in the psychology of learning mathematics.

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Research on Teaching Mathematics to the Elementary School Pupil

Jeremy Kilpatrick

Introduction

A wanderer across the deeply furrowed and sometimes exhausted terrain of educational research is surprised, if he travels far enough, at how often he encounters a sizable wilderness that has yet to be charted, let alone explored and cleared. Research on the teacher's thoughts and behavior in teaching mathematics to elementary school pupils is such a wilderness. Researchers have spent a great deal of time during the past two decades sitting in classrooms and watching teachers teach, and yet almost no one has considered how teachers approach elementary school mathematics. How do teachers conceive the subject matter of elementary school mathematics? How do they decide what instructional materials and procedures to use with various pupils? How do they decide what concepts and skills to teach, and how do they go about teaching them? How much dialogue between teacher and pupils relates to mathematical ideas, and what is the nature of this dialogue? How much teaching of mathematics is actually done by the teacher, and how much is done for the teacher by means of various instructional materials? How much mathematics is learned from the teacher, how much is learned from the materials, and what picture of mathematics do pupils take away in consequence?

Rosenshine recently noted that the describers of teaching behavior have been around for some 13 years and he could not see that we had learned anything from their efforts (IRT "Notes and News," 1977). Perhaps not. Maybe it is reasonable to turn away in confusion and disgust from the clouded "mirrors" used in studying teaching behavior, but before entirely abandoning descriptive research studies, one might consider the absurd proposition that what we need are more schemes for describing teaching behavior. More special-purpose schemes, that is.

My impression (set forth four years ago but unchanged) is that

> Most investigators of teaching behavior have tried to study as much as possible of what goes on in the classroom, and they usually devise classification schemes that are broad enough to handle almost any classroom and the teaching of almost any subject. This is understandable, since everyone would like to come up with a global theory of teaching that would fit all cases. (Kilpatrick, 1973, p.6)

I propose a more modest approach beginning with careful descriptions of a specified set of teachers teaching a specified set of topics to a specified set of pupils. For example, the teaching actions of "master teachers" explaining the concept of line symmetry to a group of bright third graders

> Could be analyzed carefully, and then compared with the teaching of the same teachers explaining a different topic to the same pupils, the same topic to different pupils, and so on. Some studies like this have been done already, but none to my knowledge have been aimed specifically at mathematics teaching, and none have included a careful analysis of the mathematics being taught.

> I recognize that the strategy I've proposed may only add to the already overblown collection of schemes for classifying teaching behavior -- at a time when others are calling for a synthesis of the existing schemes. But I feel that we have scarcely begun to investigate the teaching of mathematics per se. We need to study what goes on in mathematics classrooms from the special viewpoint of mathematics educators.
>
> Consider just one example. Suppose one were interested in studying how teachers discuss methods for solving problems. A statistic I'm fond of citing comes from a study of geometry classrooms. . ., in which less than 3 percent of all problem-solving activity was devoted to the discussion of methods. A general category system designed to describe behavior in any classroom could not include a category for such rare behavior -- one would be unable to obtain the high reliability so important to designers of such systems. But if one were studying only how teachers teach problem solving, then one could zero in on such rare behavior and study it intensively, without worrying about being able to describe in such detail all of the other classroom behavior. (p.6)

Brophy (1975) has hypothesized a qualitative change in classroom teaching that begins at about the fourth grade. Before then, he argues, there are few real class discussions, "most teacher questions are low level questions occurring in fast-paced drills, and much of the day is spent working on seatwork assignments and getting individualized feedback from the teacher" (p.12). After the third grade, however, the primary type of teacher-student interaction is probably "the kind of verbal interchange that occurs during class discussions" (p.12).

It is not clear whether this hypothesis holds for mathematics teaching. A recent questionnaire survey by the National Council of Teachers of Mathematics (NACOME, 1975) found 43 and 39%, respectively,

of the second- and fifth-grade teachers who responded saying that they used whole-classroom instruction more than half the time in teaching mathematics. The teachers reported spending an average of 43 minutes each day on mathematics instruction, with 43% of this time spent by the pupils in written or seatwork and 36% spent by the teachers in discussing or explaining. Unfortunately, these statistics are not reported for the second-grade and fifth-grade samples separately.

One of the most striking developments in mathematics education in the past decade or so has been the appearance of materials for the "individualization" of school mathematics instruction. Erlwanger (1975) has documented the misconceptions pupils can develop under an individualized program in which the instruction proceeds almost entirely through seatwork and without discussions. We do not know, however, the extent to which discussions of mathematics occur between teachers and pupils in elementary schools, nor do we know the nature and outcomes of these discussions, if they do occur. There is still room here for some useful descriptive studies.

Selecting Dimensions of Individual Difference

If all of our pupils were carbon copies, we would not need to consider the learner when conducting research on teaching. We could simply measure how much of the teaching message got through to the pupil and be done with it. With no individual differences between pupils, the importance of the pupil to research would be like that

of the computer in simulations of learning: the computer's responses are of interest, but the computer itself is not.

Individuals differ in manifold ways. A major problem for research on teaching is to determine which differences between pupils are significant. It seems reasonable to explore first those dimensions that teachers appear to consider in making instructional decisions. Brophy and Good (1974) have reviewed the literature on some of the more important dimensions of pupil difference.

Researchers and teachers may differ as to the importance of a dimension of pupil difference. In particular, researchers may undervalue a dimension if they consider the associated differences in learning all but impossible to influence. A good example of such a dimension is the sex of the pupil. Although sex differences in classroom interaction patterns have been studied extensively (see Brophy & Good), researchers in mathematics education have often treated the pupil's sex as essentially irrelevant to mathematics teaching and learning. In the National Longitudinal Study of Mathematical Abilities (Wilson, Cahen, & Begle, 1968-1972), the achievement of pupils who had used different textbooks in learning mathematics was analyzed separately by sex, but the data on sex differences in achievement and interactions between sex and textbook group were put into the appendices to the technical reports. The argument heard at the time was that sex differences in mathematics achievement may be interesting to a handful of researchers, but such differences are not really subject to manipulation by the teacher or the textbook writer. Interactions between sex and textbook group

were dismissed because no one could seriously imagine publishers producing separate mathematics textbooks for boys and girls.

The question of sex differences in the learning of mathematics has become a more prominent research topic in the last few years, however. Researchers concerned about the large and apparently growing number of girls who enter college handicapped by failure to take mathematics courses in high school have turned their attention to the roots of this failure. One can hypothesize that, relative to other factors, the teaching of mathematics in elementary school has little to do with the sex differences in patterns of mathematics course-taking that occur in high school. This hypothesis has not been tested, however. The newer argument is that if one could document how elementary school mathematics teaching contributes to the diversity in mathematics learning of male and female students, then even though this contribution might be relatively small, one could bring it to the attention of teachers and others, thereby initiating some remedial action.

I raise the issue of sex differences to make the point that researchers should consider carefully their reasons for selecting a dimension of divergence for study. The sex of the pupil has often been used as a variable in research because it is easy to measure and because other researchers have found sex differences to exist. Yet, even though mathematics is a subject in which some of the largest and most consistent sex differences in achievement have been found, researchers in mathematics education have usually been reluctant to take sex differences as a focus for their work. They have

argued that one should select individual difference variables whose effects are easily influenced by teachers and curriculum developers. Nevertheless, ease of measurement, appearance in previous research, and ease of manipulation are only three criteria to consider. Other, more important criteria are relevance to educational theory, salience to teachers, and importance to society.

Modeling the Interaction Between Teaching and Pupil Differences

Although the aptitude-treatment interaction (or ATI) model (Cronbach, 1957) is ultimately an unsatisfactory way to express the relation of teaching and pupil differences to each other and their joint relation to learning, it does offer a first approximation. In its classic formulation, the ATI model uses the differential correlation between an aptitude variable and a payoff variable under two or more treatments. For the sake of argument, take any dimension of pupil differences as the aptitude variable, any desired learning outcome dimension as the payoff variable, and any set of teaching decisions manifested in behavior (call it teaching strategy) as a treatment. Then, if the teaching strategy having the most favorable outcome is different at various points along the pupil difference dimension, the ATI model theoretically allows one to assign a pupil to the teaching strategy that yields a maximum payoff.

The typical procedure in ATI research has been to select or construct treatments that differ along some dimension, consider the treatments as relatively fixed, and then search for aptitude

variables that might yield an interaction. Such searches have often been fruitless, perhaps in part because researchers have supposed that individual difference variables that are positively correlated with learning will make good candidates for an ATI -- a supposition that confuses main effects with interaction effects.

> Suppose, for example, that one had observed that field independent pupils . . ., under a variety of instructional strategies in mathematics, tended to learn more than field dependent pupils. It would appear that the more this relationship had been observed across situations and variations in strategies, the harder it would be to devise a strategy under which field dependent pupils would achieve relatively more and field independent pupils relatively less. (Kilpatrick, 1975, p.74)

One can argue that a more profitable procedure -- especially in research on teaching -- would be to postulate an infinite set of treatments that vary along some dimension. Instead of picturing two or more regression lines in a plane, one can picture a three-dimensional surface whose intersections with parallel planes can be projected to yield regression lines. Actually, one can do away with the regression lines -- which portray information about groups, not individuals -- and simply visualize a swarm of points in a three-dimensional space defined by pupil difference, learning outcome, and teaching strategy variables. Each point represents a hypothetical pupil under a fixed strategy.

Now, suppose a situation in which a teacher -- attending to one dimension each of pupil difference, learning outcome, and teaching strategy -- uses something like a mental image of such a swarm of points when he teaches. He is faced with a set of pupils; he knows something of their individual differences, something of their learning

outcomes under previous instruction, and something of the teaching strategies used in previous instruction; he uses the image of the swarm of points to estimate where his students are likely to be if their subsequent learning is like their prior learning. He then can determine how he might vary his strategy to increase their learning.

Even this extension of the ATI model is, of course, a gross simplication -- and a possible distortion -- of reality. As noted earlier, pupils differ in numerous ways. Teachers typically expect multiple learning outcomes. Words such as "treatment" imply a uniformity that is plausible only if one does not go into classrooms to see how teaching actually proceeds. All of these variables undoubtedly interact in a complicated fashion. If a teacher had a mental image like the one postulated, it would have to be capable of adjustment as his attention turned to different dimensions of individual difference and learning outcome. Furthermore, it is not at all clear how one might dimensionalize teaching strategies (although somehow movement along a continuum seems a better model for much of what teachers do than selection from a small sample).

As a second approximation, however, the extended model does show the researcher that instead of looking first for an individual difference variable that could yield an interaction, a better strategy for research on teaching might be to study a teacher deemed likely to be effective as he teaches pupils deemed likely to have some difficulty in learning. To what dimensions of pupil difference does he attend, and how does he decide to modify his strategy?

Can he predict which pupils are likely to fail and which are likely to succeed? If one assumes that capable pupils are likely to succeed under a variety of strategies, then a study of the teacher's success or lack of success with the less capable pupils could yield some insight into various individual difference variables that might affect learning. By looking at how the skillful teacher adjusts instruction to various pupils, one may uncover individual difference variables that have some organic relation to the learning process -- variables that will be closer to the "new aptitudes" that Glaser (1972) has postulated.

The ATI model has been criticized as too narrow. Hunt (1975) has argued that the broader notion of the B-P-E (behavior-person-environment) paradigm should be used for studying interactions. He says that "psychologists, especially educational psychologists, must eventually face the challenge posed by the complexities of person-environment interaction, and the B-P-E paradigm permits conceptualizing these otherwise impossibly complex problems in a form that may lead to understanding" (pp. 227-228). I have tried to suggest how the B-P-E paradigm might be used in research on teaching mathematics to the elementary school pupil, and how one might begin by testing whether teachers operate under some such paradigm.

Identifying Pupils' Mathematical Abilities

The research literature offers some clues regarding individual difference variables that influence the learning of mathematics.

Constructs such as general mental ability, reasoning ability, developmental level, cognitive style, and learning style have been proposed as candidates. How these variables might operate to influence mathematics learning is not obvious. To find out, one apparently needs to go beyond gross differences in scores on measures of these variables. I propose, as alternatives, some variables that are more molecular than these, that seem more closely related to differences in mathematics learning, and yet that teachers can reasonably be expected to observe in their pupils. The variables I propose are the mathematical abilities identified by Krutetskii (1976).

Krutetskii uses an information-theoretic model of mental activity in mathematics. He postulates three stages in solving a mathematical problem: (a) gathering the information needed to solve it, (b) processing the information so as to obtain a solution, and (c) retaining information about the solution. To each stage correspond one or more abilities, yielding a structure that can be outlined as follows:

1. Obtaining mathematical information

    A. The ability for formalized perception of mathematical material, for grasping the formal structure of a problem.

2. Processing mathematical information

    A. The ability for logical thought in the sphere of quantitative and spatial relationships, number and letter symbols; the ability to think in mathematical symbols.

    B. The ability for rapid and broad generalization of mathematical objects, relations, and operations.

C. The ability to curtail the process of mathematical reasoning and the system of corresponding operations; the ability to think in curtailed structures.

D. Flexibility of mental process in mathematical activity.

E. Striving for clarity, simplicity, economy, and rationality of solutions.

F. The ability for rapid and free reconstruction of the direction of a mental process, switching from a direct to a reverse train of thought (reversibility of the mental process in mathematical reasoning).

3. Retaining mathematical information

A. Mathematical memory (generalized memory for mathematical relationships, type characteristics, schemes of arguments and proofs, methods of problem-solving, and principles of approach).

4. General synthetic component

A. Mathematical cast of mind

These components are closely interrelated, influencing one another and forming in their aggregate a single integral system, a distinctive syndrome of mathematical giftedness, the mathematical cast of mind. (Krutetskii, 1976, pp.350-351)

Krutetskii emphasizes that this outline refers to the mathematical abilities of a schoolchild, and not necessarily to those of a mature mathematician. He argues that the specific content of the structure of abilities depends largely on how the pupil is taught, but that the component abilities listed in the outline will always be present in some form.

Consider how the abilities identified by Krutetskii might be used in research on the teaching of mathematics to elementary school pupils. Depending on the research question to be pursued, a specified

sample of pupils and teachers would be chosen. For example, some teachers might be chosen who were reputed to be outstandingly effective in teaching mathematics. Data on the thoughts and behavior of these teachers as they plan, conduct, and evaluate mathematics lessons could be gathered by such means as interviews with the teachers, videotape recordings of the teachers' activities, and interviews with the teachers as they view recordings. Analyses of these data might be directed toward the identification of pupils perceived differently and treated differently during the lessons. Information would have been obtained before the lessons as to the pupils' mathematical abilities. For example, the ability associated with the process of obtaining mathematical information -- the ability to grasp the formal structure of a problem -- might be assessed by asking the pupils to find structural similarities between different problems (Silver, 1977). One might begin with the hypothesis that pupils who differ in various information-processing abilities will be perceived and treated differently as a function of the abilities required in the tasks presented in the lessons. Or, one might explore the relative salience of these abilities as opposed to other attributes of the pupils. Or, one might reassess the abilities to trace how teaching had changed the abilities of various pupils. Krutetskii's outline, coupled with an analysis of the information-processing demands of instructional tasks, provides a framework for studying how teachers adapt their teaching to differences in pupils' mathematical abilities. A natural extension of this line of investigation would be to study the effects of providing teachers with feedback about

their pupils' abilities.

Making Decisions About Teaching

In a fascinating little book that is not well known in this country, Bishop and Whitfield (1972) offer some resource materials for teachers to use in practicing decision-making skills. Actual teaching situations -- all of a "critical incident" nature -- are presented "chiefly to give student teachers practice in thinking about appropriate on-the-spot decisions, under the guidance of their tutors" (p.v). Bishop and Whitfield offer a scheme for classifying situations according to "areas" (learning, relationships, and environment) involved in the implied decision. For example, the following situation involves primarily pupil/pupil and pupil/adult relationships:

> In a group of three girls working on a project, one girl is dominating and not letting the others help at all.
>
> What action would you take, if any? (p.13)

More important than classification of the incidents, however, is the way in which they are to be used. Bishop and Whitfield suggest that the situations be presented in a discussion group with several student teachers, that the situations be discussed in detail (with emphasis on possible causes, available information and options, and decision criteria), and that additional situations then be presented under some time pressure. In each case, each student teacher is expected to choose a "best" course of action even though there is obviously no universal "best" course.

As another sample situation, consider the following:

> John is 9, and says that in subtraction you take the smaller number away from the larger one. He does the following subtraction like this:
>
> $$\begin{array}{r} 347 \\ -268 \\ \hline 121 \end{array}$$
>
> What would you do about this? (p.35)

Questions that might be used in discussing the situation with student teachers include: (a) What are some of the alternatives open to you? (b) Which alternatives are likely to be most successful? (c) What criteria did you consider in answering (b)? (d) What additional information might you want to obtain?

Although such situations can be used effectively in teacher education programs to make student teachers aware of alternative options and criteria, I would like to consider their possible utility in divising instruments for research on teaching. I am aware that there exist simulations of teaching situations that are available in writing, on film, or in "multi-media" form, but I am not aware of simulations that have been constructed to explore, more or less systematically, the decision rules teachers use in teaching elementary school mathematics. I am especially concerned here with how teachers diagnose and attempt to correct errors and misunderstandings. The reason for using Bishop and Whitfield's approach is to focus attention not only on the decision the teacher actually makes, but also on the set of alternatives and criteria he considers. Shulman and Elstein (1975) have suggested a variety of ways one could investigate the

decision rules used by the reading diagnostician and the classroom teacher. It does not take much imagination to transfer these suggestions to the teaching of elementary school mathematics, but perhaps it will take some thought to come up with tasks that simulate key situations in mathematics teaching.

Adapting Instruction to Individual Differences

There is little point in considering learner variables in research on mathematics teaching unless one examines how such teaching is adapted to individual differences. Every teacher makes some attempt to adjust his teaching to the characteristics of his students, but some teachers are more adept than others at perceiving relevant characteristics and making appropriate adjustments. "Master teachers" and "teachers deemed likely to be effective" were mentioned earlier as possible subjects for research because of an implicit belief that the source of some of their effectiveness must lie in their sensitivity to individual differences in abilities, in general, and in mathematical abilities, in particular. More than a decade ago, Hawkins (1966) observed that rather than study average teaching, one ought to locate good teaching situations -- say, the best one percent -- and study those. Just as chess masters and expert clinicians have been used in previous studies of judgment and decision making (Shulman and Elstein, 1975), so master teachers of mathematics ought to be used in research on mathematics teaching as clinical information processing.

Simon and Simon (1962) found that grandmasters of chess differ from ordinary players not in memory, processing speed, or insight, but in their use of powerful heuristic rules to guide their selection of moves. By analogy, one is led to ask whether master teachers differ from ordinary teachers in their use of heuristic rules to guide their selection of teaching moves.

I once had the good fortune to observe and assist for several years an undisputed master of the teaching of mathematics: George Polya. I observed not only his teaching, but also his preparations for teaching and his evaluation of his teaching in dialogues with me. I was struck by his perception of the capabilities of the students in his classes and by his ability to "replay" a teaching episode in his mind, noting what he might have done differently. I think he possessed powerful heuristic rules for deciding what question to ask a student or what example to use to illustrate a concept. Most of these decisions were made during his planning for the teaching episode, but some were obviously made on the spot. I have seen him baffled by a student's response and searching for a question that would reveal the student's thinking. I have watched him capitalize on another student's offhand remark, revealing an unexpected dimension of the problem under discussion. He often struck me as being like an orchestra leader, following a score, but getting a unique rendition from the players before him.

There must be master teachers of elementary school mathematics who would permit us to study how they plan, conduct, and evaluate their teaching. We could assess their knowledge of their pupils'

mathematical abilities and map their rules for making instructional decisions. We could study the dimensions of pupil difference they consider in making these decisions. We could compare their decision rules with the rules used by ordinary teachers.

The outcomes of this research could be applied in at least two ways: (a) to create instruments for assessing a teacher's decision-making skills, and (b) to study the effects of feedback about their decisions on teachers' subsequent decisions. The first application could be done in straightforward imitation of previous work in evaluating medical problem solving (Shulman & Elstein, 1975), and for a similar purpose. People who make competency-based evaluations of teachers might be happy to have an instrument that would have a link to teaching practice. The second application seems even more promising. Brophy (1975) is optimistic about the effects of feedback: "When teachers are acting in self-defeating ways without being aware that they are doing so, they very quickly change their behavior if the problem is called to their attention" (pp. 2-3). Part of the trick seems to be to demonstrate that their behavior is not in accord with their beliefs about effective teaching.

The remarkable popularity of "individualization" to describe what one is trying to do in an elementary school mathematics class suggests that it is more than just an educational fad. Of course, "individualizing" may consist of nothing more than repeating the same instruction for those pupils who did not master it the first time, or it may be essentially the same instruction presented at different rates. The available instructional materials certainly

do little to inspire confidence that learning "styles" have been identified and instruction matched to them (NACOME, 1975). Although I think we should continue to search for consistent patterns of individual differences in mathematics learning, I am not sanguine about the identification of learning styles. However, I do think "individualization" itself is here to stay because it expresses a basic consideration in teaching. Teachers worry about individualization because some pupils are not as successful as others in learning. Teachers need to diagnose learning failures in order to decide how to remedy these failures, and this process of diagnosis requires some conception of the child's mathematical abilities.

Ginsburg (1977) has emphasized the importance of informal interviewing in diagnosis. He finds it next to useless to think of children's errors in terms of "low IQ," "low mathematical aptitude," or "learning disabilities." He stresses instead knowledge of the specific roots of mistakes: failure to align digits or faulty memory of the addition facts. I have put forward Krutetskii's model of the structure of mathematical abilities because I think it may lead to an intermediate level of theory that could be of help to the teacher in making diagnoses and, perhaps more important, in dealing with classes of errors when a detailed individual diagnosis is not feasible.

Part of the process of diagnosis will always depend on the art of the teacher. Timmer (1977) has had some success in getting secondary school mathematics teachers to use diagnostic procedures by providing them with an "item bank." Since the teacher must pick and

choose items as well as write items of their own to come up with a comprehensive instrument, Timmer has termed the item bank a "half-product." He argues that half-products are an important means of communication between researchers and teachers. Half-products ensure adaptation to changing conditions, they enable teachers to participate in product development, and they are more acceptable to teachers.

Rather than seeking "teacher-proof" methods and materials for diagnosis, perhaps we should look toward half-products. Research studies of the sort discussed in this paper could lead to half-products for teachers to use in examining their pupil's mathematical abilities, their own teaching, and the relation between them.

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A Reaction by Jere Brophy*

In general, I think that this is a well reasoned and useful paper that presents many thought-provoking research questions. Obtaining the answers to these questions would probably substantially improve mathematics instruction in the elementary grades. The questions are also of the kind that should be of interest to the IRT and others interested in clinical information-processing approaches to research on teaching. My remarks are ordered according to passages in the paper distributed at the conference.

Kilpatrick calls attention to the need for research on the teaching of mathematics which takes into account the subject matter and teaching techniques unique to that subject. I agree that this needs to be done in mathematics as well as in other subject areas. Doing it well would not involve redundancy with existing instruments or research, because the kinds of observations that would be needed to do the job properly would be very different from systems typically used, which stress frequency counts of very general kinds of behavior. Instead, methods would be tailored to the measurement of much more specific kinds of teaching, specific not only to mathematics as a content area, but to the particular teaching situation being studied (e.g., introducing

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a completely new concept or skill; introducing a new objective that depends upon previously learned objectives; conducting recitation and giving feedback; explaining assignments; monitoring seatwork and reteaching, etc.). Such systems should not only specify the general kinds of teacher behavior that would be required, but in particular should specify criteria of quality or appropriateness. This would allow observers to code not merely what the teacher was doing, but how well.

This enterprise also would involve solving some of the outcomes problems mentioned repeatedly during this conference. The outcome criteria for such research would not be long-term performance on standardized tests, but short-term outcomes such as the percentage of students who understood an explanation once it was given, the percentage who were able to meet certain objectives in seatwork assignments (or perhaps the percentage who successively met more difficult objectives), the degree to which the teacher had to repeat or reteach points to make them clear, teacher efficiency or level of success in identifying, diagnosing, and remedying student confusion and error.

Concerning the comments Kilpatrick made about the process of mathematics instruction as opposed to the process of instruction in other subject areas, I agree that the more important switch in mathematics is from low level questions and drill toward individualized written work, rather than toward discussion. This means that, whether or not math is unique as a subject area, it certainly is different from the discussion-based instruction which has been the focus of most research on teaching. Consequently, the accumulated body of knowledge about research on teaching, undifferentiated according to subject matter, may

have been collected mostly in subject matter areas other than mathematics and may thus generalize less successfully to mathematics. This becomes yet another reason for the kind of specific attention to mathematics instruction called for by Kilpatrick.

As to the comments about sex differences, it is true that sex differences are often studied because they are easy to investigate. It is also true that teachers and curriculum writers cannot do much to change the sex typing that reflects the values and influences of society at large; however, they can tailor materials and instruction to the specific needs and interests of learners. For one thing, mathematics has clearly been identified in this society as a subject of interest to males. Males are expected to do better than females in math, and to like it better, as well. This view could be counteracted directly with systematic, conscious attempts to stress female models and characters in books about math and in word problems in order to make math relevant and interesting to females.

Research by Stein, Pohly, and Mueller (1971) has shown that when contexts of problems were presented as relevant to the female sex role, rather than to the male sex role, girls found them easier to solve. Not only were aspirations and expectations for solving these problems higher (though mathematically the problems were identical), but the problems themselves were actually easier to solve. It probably <u>is</u> absurd to think about writing separate textbooks for each sex, but findings like these and the generally masculine image that math has had suggests that a more systematic attempt to make it a subject relevant to females would be worthwhile.

Concerning more general comments about other aptitude treatment interactions (ATIs), I must confess an uncertainty similar to that expressed by Kilpatrick. The search for ATIs has had a checkered history. There was great interest in them for a long time, but research results were discouraging. Just when it looked like everyone was about to give up, however, researchers began to report ATIs regularly, and more and more are coming out everyday. However, it is as difficult as ever to try to determine what this may mean for curriculum development and instruction. First, most so-called ATIs really concern student personalities and preferences rather than aptitudes, and and it is often neither feasible nor advisable to try to arrange instruction to give all students what they prefer. Also, as Kilpatrick has pointed out, most interactions that are obtained are not disordinal interactions, suggesting that one treatment is best for one type of student and another treatment is best for another type. Usually, the interactions that are obtained indicate that one treatment works best with a particular kind of student, but also as well with all kinds of students, and that another treatment works well only with one type of student (particularly the brightest ones). Although such data are technically statistical interactions between treatments and students, the implication is usually the same as the implication would be if main effects had been observed: one treatment is better than the other because it is more effective generally.

It may be, however, that the treatment that works well with the brighter students is much shorter or more economical, in which case it might be worth using for those students so that they can move on to

something else. The logical extension of this kind of reasoning leads to notions of individualized instruction rather than ATI, however. Proponents of ATI usually assume that goals of instruction are the same -- only the methods differ. However, the situation discussed here involves different instructional goals for different kinds of students, not merely different strategies to reach the same goal. The implication is that brighter students might be taught more objectives or different kinds of objectives, not merely that they would be taught differently to reach the same objectives.

These comments apply as well to the three-dimensional model used by Kilpatrick in discussing ATI and related concepts. This model and its apparent implications seem to fit very nicely into the mastery learning approach articulated by Bloom (1976). The contrasts between the extremes discussed in this three-dimensional model are very comparable to the contrasts between traditional and mastery learning approaches as described by Bloom. I recommend Bloom's book to mathematics educators who might be interested in the reasoning behind individualized instruction. In contrast to most other approaches, Bloom's model is not tightly tied to programmed instruction or some particular model of individualized instruction, nor does it assume particular curricula or methods.

Kilpatrick makes many good suggestions for research in the section on ATI s . Again, these suggestions seem likely to yield useful information, and many of them (especially those involving identifying outstanding teachers and studying how they modify their strategies to take pupil needs into account) seem particularly important for consideration

by the IRT. Concerning these suggestions, I agree with the statements made by several individuals at the conference that we need to switch our attention from "master teachers" to "masterful teaching," indicating recognition that good teachers are not outstanding in everything or outstanding all of the time, and that what constitutes successful teaching needs to be defined in terms of its immediate and long term outcomes. As a compromise, I think that it might be possible to identify outstanding individuals who are "specialists" at particular tasks of teaching. Some teachers are consistently very good at explaining concepts clearly, others at encouraging students to understand (vs. merely learning to give the answer), and so on. The concept of algorithms of teaching as presented by Landa may be useful here in conceptualizing the aspects of teaching to be isolated for study.

I would also elaborate upon the suggestions of Kilpatrick by noting that attempts by myself and others to study teacher individualization of instruction in whole class settings have not worked out very well. It is difficult to get enough data about each individual student in an entire class to allow the researcher to perform the analyses he/she would like to perform. There are just too many students, and not enough is known about their unique situations to generate ideas about what appropriate individual instruction might mean for each one. However, I think that studies like this can be done very successfully when teachers are working with small groups of four or five students -- ideally, groups composed of individuals with clear but contrasting learner characteristics. Studying teachers interacting with several such groups would allow the researcher to tease out systematic contrasts in treat-

ment of different kinds of students from other differential behavior that is specific to situations or to the particular students involved, not really indicating a teacher attempt to tailor instructional strategies to particular needs. Information from studies like this could yield valuable information in answer to the questions raised in the first half of Kilpatrick's paper.

In a section titled "Identifying Pupils' Mathematical Abilities," Kilpatrick elaborates the ATI paradigm to suggest questions specific to mathematics instruction in the elementary school and ideas about how research on them could be conducted. He tentatively suggests Krutetskii's model of mental ability in mathematics as a basis for classifying students. This might be useful, but I believe that more useful results will come from studies based on classification of tasks rather than students, at least for initial classification. That is, I think that it makes more sense to identify key teaching tasks (or algorithms of teaching) and to study teachers performing these tasks with different kinds of students than it does to identify types of students and study teachers doing a variety of things with these student types.

Typing of students would soon become relevant, however, because different students would bring different degrees of ability into these sessions and would behave differently during them. Consequently, it would be important to include strategy and error analyses to identify student action tendencies in response to the demands of different kinds of mathematical tasks; it would be important to know how teachers responded to the particular student needs that these different problem solution strategies and patterns of misunderstanding and error would

present. My own belief is that knowledge of this sort constitutes the most important kind of knowledge about what is involved in teaching mathematics.

The ideas in this section of Kilpatrick's paper strike me as very compatible with the CLIPIR project at the IRT. This project uses information gleaned from a variety of diagnostic reading tests (along with other information) as stimulus input in studies of the diagnostic and remedial activities of skilled reading clinicians. It strikes me that many of Kilpatrick's suggestions in this paper could lead to parallel work in mathematics instruction. This is also true about his suggestions for research based on the critical incidents of Bishop and Whitfield (mentioned in the section, "Making Decisions About Teaching").

Concerning Kilpatrick's comments on "Adapting Instruction to Individual Differences," I still am very optimistic about the usefulness of feedback to teachers as a way to improve their teaching; I believe that this applies as much to mathematics instruction as it does to any other kind of instruction. Feedback will increase teachers' awareness of what they are doing when teaching, and why they are doing it. This means that instruction in parallel situations will become more sophisticated. Unsystematic reactive instruction will become more systematic and proactive, being directed by what amounts to algorithms. Further, what was systematic instruction (the following of algorithms without deeper understanding) or systematic behavior (when the algorithms did not work) will become a more elaborated, flexible, and essentially rational activity. It should be noted, however, that this assumes the goals of instruction and the associated curriculum materials to be sensible in the first place. Even optimal teaching will not succeed when this

is not the case. (I have serious reservations about many of the instructional goals that mathematic educators propose as appropriate for elementary school mathematics instruction.)

I believe that Kilpatrick's notion of "half-products" is a good one. Along with the provision of systematic and useful feedback, this strikes me as an important component of inservice teacher education that would capitalize on good preservice education and development and dissemination of good materials. I would encourage anyone who has an opportunity to influence such things to take action to include much more explanation and instruction in the teacher manuals that accompany curricula used in schools. It is very inefficient and unrealistic to assume that whatever teachers have learned about mathematics and mathematics instruction in college courses is going to be retained, generalized, and applied later to the specific mathematics instruction of their pupils. This practice will not eliminate the problem of teachers' misunderstanding the objectives of particular lessons and subverting them through inappropriate teaching. However, probable sources of confusion and error could be minimized by including appropriate explanation, special instruction, or cautions about things that could go wrong or which must be noticed and remembered. Having done some of this myself in preparing teacher manuals and curricula, I know that it can be difficult to do but that the payoff is very high. Teachers appreciate and use the added information, and it does succeed in minimizing misunderstanding and inappropriate instruction. Furthermore, it forces curriculum developers to think seriously about what instructing each lesson will mean in practice, something that forces revision of lessons more often than most of us would like to admit.

Summarizing this final point, I would say that although we should respect, expect, and use practitioner wisdom, and although we should not attempt to make teacher-proof materials, we should provide teachers with clear and complete information about the purpose of each lesson (not merely the immediate objective, but the larger purpose in terms of developing mathematical insights and ability), how to teach the lesson, and what to do when certain predictable difficulties occur.

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A Reaction by Leo Landa*

Kilpatrick's paper touched on a number of important questions about teacher training research. I agree with his criticism of current descriptive methods of analyzing teachers' activities and his suggestion that a restructuring is in order. Choosing relevant parameters is essential for effective analysis. As Kilpatrick indicates, much of the literature on teacher training research deals with parameters in isolation; it is not clear which ones, either alone or in combination, influence instructional outcomes.

It is also evident that insufficient attention has been given in the teacher training curriculum to the teaching of methods of thinking. One of the most important problems in research in this field is how to describe teachers' mental behaviors. This is a key point because in order to teach teachers to teach, it is necessary first to understand their thinking processes. Instead of abandoning descriptive research studies entirely, we might work out more general methods of description in order to make such studies more easily transferable. Kilpatrick calls these general methods schemas. I prefer the term "algorithm" because the descriptive method really breaks down the process of teaching into its constituent operations. A productive way to describe both the external behavior and internal decision making of teachers is

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to break down cognitive operations into elementary components. Of course,not all problems are of an algorithmic pattern. For this reason it is not possible to completely algorithmize a given teacher's activity. I propose, however, that the analysis of typical didactic problems which exist in mathematics teaching will enable us to write algorithmic solutions for such problems, then teach teachers to use them. There may exist a series of algorithms, but if we identify this series, teaching our students to use it, they will be able to function in varying situations, relying on reasoning patterns they learned from algorithmic solutions. I am advocating the use of algorithms of teaching as an operational means of training teachers how to reason pedagogically.

Another interesting and important point Kilpatrick makes is that teaching methods should be tailored to specific abilities. The problem lies in which abilities to choose, which abilities to isolate as "competent" or "constituent" abilities. For instance, one ability classed by Krutetskii as a competent ability is for obtaining mathematical information -- the ability for formalized perception of mathematical material (grasping the formal structure of a problem). I am convinced that this is not an elementary ability. Furthermore, it is not possible to teach or to develop this ability unless it is analyzed into more elementary competencies.

In the first section of my talk I spoke about algorithms of teaching, that is, algorithms of teachers' cognitive and practical activity. Efficient instruction on an algorithmic basis presupposes that algorithms are not handed to students in ready-made form, but are taught in a way that requires students to independently discover the

form. How to teach in this way is a heuristic, creative problem. That's why algorithmic and heuristic processes are not separate; there is a very close interaction between them.

One of the substantial differences between algorithmic and heuristic prescriptions is that algorithmic prescriptions determine the solution process completely, while heuristic prescriptions always contain some degree of uncertainty. We can teach heuristic operations and algorithmic operations in a similar way although they are quite different types of teaching.

I agree with Kilpatrick's comments on teaching through the use of critical situations or incidents. Presenting students with critical situations is extremely important but is not sufficient to train a teacher. The key idea here is to not simply confront a prospective teacher with a critical situation, but to teach him/her how to reason in such situations.

I quite agree that it is important to discover the internal mechanism of a teacher's decision making process. Yet, to do this we need more than interviews. We need purely objective methods which permit us to discover the regularity of teachers' reasoning without overtly asking them how they reason. It is important to consider other methods of arriving at internal processes because many times an expert in a field is simply not able to explain why it is he/she arrived at a given solution. Finally, I concur with Kilpatrick on the importance of individual differences and adapting instruction to these differences; but how is it possible for a teacher to take into account the individual psychological aspects of each pupil when he/she has to deal with dozens

of students simultaneously? While I agree with the necessity for individualization of instruction, I still question how it can be done.

Discussion*

Byers: Professors Landa and Kilpatrick have indicated that teachers may teach using S-algorithms and still produce heuristically-based concepts in children. Esty counters that teachers must teach heuristically if they are ever to produce heuristic learning and understanding in children. Which position are you more comfortable with, Dr. Davis?

Davis: I don't consider the two views to be opposed. What kids learn and ultimately use isn't necessarily what a teacher puts into their heads. I think, however, that if more teachers thought heuristically, more kids would do better. David Page has said that the name of the game in teaching the "new" math is to help all kids learn math the way successful kids are learning it. Some kids develop the right heuristic procedures no matter what the teacher does. Simply using more examples when teaching math might induce kids to see and manipulate the information in novel ways, resulting in heuristic learning.

Kilpatrick: I think we're using the terms "heuristic" and "algorithmic" in several different ways here. It's important to distinguish between heuristic teaching (the teacher's use of heuristics to decide what to do as a teacher) and the development of heuristics on the part of the kids. It also seems to me that teaching is too complicated to allow us to pick out a set of so-called "critical tasks" a teacher must perform and label them algorithms.

In our teacher education programs at the University of Georgia, we

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*This written discussion, based on audiotapes of the conference, has been condensed and paraphrased to include only key points.

spend a lot of time with algorithms, but we don't spend much time with what I see as a very large part of what teachers are occupies with -- decisions about what to do in a particular situation with a particular set of kids, decisions made on a heuristic basis. I think it's better to stress the heuristic things teachers do rather than the algorithmic things they do.

Landa: The decision of whether to use heuristic or algorithmic methods of teaching cannot be made arbitrarily. It is necessary first to decide objectively which problems given to students to solve may be solved algorithmically and which may be solved non-algorithmically.

Shulman: So-called "objective" task analysis, however, might be contingent on a subjective definition of what the curriculum surrounding the task consists of. If you have a certain conception of the curriculum as being a developmental or longitudinal series of tasks, you may well analyze a specific task differently than you would if you did not have such a conception.

Lesh: There's an additional problem here, too, that of connecting the analyses with actual teaching. Analyzing the math problems, the students, and the teaching "moves" will do no good without an intuitive teacher who can put all the analyses together to use in particular situations.

Shumway: We need to discover some organizing concepts to apply to our data, some set of unifying elements which would allow a teacher to behave in a certain way almost intuitively -- based on his/her underlying structure of concepts.

Lewis: I'm concerned, too with the fragmenting that occurs when studying the teaching process. I'm currently working on a framework which involves the notion of script. A script is a way of conceptualizing or representing the system of action that is carried out in a context over time. The aim is to look at the classroom context not as static, but as a dynamic flow of episodes. Looking only at the teacher(or only at one pupil), we apply an information-processing model to try and understand the "script" that person is following.

On any given day, the teacher enters a situation with an initial state of knowledge-- his/her way of representing the world and the classroom environment, his/her current assessment of each learner, the goal for that day or maybe the goal for a topic. The teacher has certain attitudes towards the students, the educational process, the classroom, etc. All these things enter into the decision of the teacher as to what actions to take and the evaluation to make as a result of those actions. Perhaps a student asks a question; the teacher has to respond in a certain way. The response the student has made (asking the question) brings about a great deal of teacher evaluation with respect to the goals the teacher has set. There is very complex interaction going on -- picking up on students' responses, on blank faces, on whatever is occurring in that situation.

I think this notion of a script is akin to what Landa is calling an algorithm except the script goes a little deeper. It doesn't hinge on the subject matter or on instructional strategy, but includes components of each.

Ashlock: Is the script located in the teacher's head?

Lewis: Yes, it's a mental thing.

Ashlock: So this is a metaphor for describing how teaching proceeds.

Kilpatrick: Right, like Bellack's metaphor of classroom interaction as a game -- though in this case you are looking at the ideas that the teacher has, not just at the moves that are made while playing.

Lewis: Yes. I'm not just looking at behavior; what I'm trying to do is account for why the behavior occurs. Ultimately, I'm interested in prediction.

The script would also be affected by other things such as parental conferences or whether the teacher burnt the toast this morning. Presumably what we want to assess, however, are regular patterns. There are patterns for general scripts which are followed. I certainly would want to allow for the teacher's ability to change a script that isn't working. The script is not like a tape that you put on a tape recorder and let run -- the script can be changed. In that sense, the teacher is a playwright.

Davis: Did you get the original idea for the script from Roger Shank?

Lewis: Yes, Shank and Avison.

Davis: This idea was originally concocted by the information-processing school to explain how a computer can load the correct tape. Since a person can't search all of memory for something, he/she has to have some way to grab only the appropriate parts. Shank would say, "What is it you are going to load? You are going to load this script because you know you are in a grocery store, you know you went there to buy groceries, etc."

Lewis: There's another motive for choosing the script idea, too. I feel that the classroom environment is a social situation -- something is taking place over time, resulting in the need for a dynamic model.

Davis: I am terribly taken with this idea that you load something in your head and you work from that, under certain circumstances revising the whole thing. Still, someone needs to explain how a person can cast some data aside or even wipe all of it out.

Esty: You're saying that there is some sort of superstructure that elects the script.

Lewis: I don't know. I think it may be a question of the context selecting the script.

Shumway: As an example, take a chess master. He looks at the chess board and walks away, knowing exactly where the pieces are. He doesn't do it by remembering where every piece is; he just sort of "knows" where they are. Different people see different things because they attend to different things. It's not just a matter of casting aside things that don't matter. Even in a chess game there is an awful lot that does matter; the chess master simply doesn't see it all.

Byers: I have a hunch that he does see it all; he just may not be aware that he sees it. It's like what we were talking about this morning -- the speed with which a kid will multiply 3 x 15. To ask him to analyze what went on is impossible. The act has become a single functional unit.

Highly skilled artists have a much more difficult time describing how

they do something than people who are not nearly as competent. Eventually you get to the place where you can't verbalize the steps in a process; it's all integrated into a single unit. I think the point that Lewis made about throwing out irrelevant information is a very good one. It makes perfectly good sense that a person who is well skilled in an area has learned to identify what is relevant and what is irrelevant.

Shulman: The only way to understand how incredibly complex stimulus events result in only one of many complex responses is to tie the events to a rather simple set of internal propositions the teacher must have.

An example is the work that we did with analyzing the responses of reading diagnosticians. It took two or three cues for a proposition to generate one or more working diagnostic hypotheses about what was occurring in a given case. The generation of the hypothesis drew from memory a subroutine carrying with it an inquiry and a set of operations associated with confirming, varying, or misconfirming the hypotheses.

Each hypothesis was not unique simply because there are limits on the kinds of problems a reading diagnostician will encounter. I suspect the same is true in teaching mathematics. One of the things that distinguishes the clinical training of physicians from the training of chess masters is how many games each has a chance to play. That is, how much variety each experiences in order to build up a memory for hypotheses or "scripts". In chess, systematic disciplined practice is an indispensable element. Yet, how many different experiences in the actual teaching of mathematics does an elementary education major get? It is remarkable how little specific practice any student teacher gets.

I think there needs to be some simplification of the information-processing model. We did some work with information-processing behavior in medicine here at MSU -- designing experiments in which we trained young physicians in cognitive processes which we believe are those which enable the generation of many hypotheses. Ultimately, we may need to emphasize not training research but curriculum research to create a teacher education curriculum that represents these approsches.

ENVIRONMENTAL ISSUES IN THE TEACHING OF MATHEMATICS IN THE ELEMENTARY SCHOOL GRADES: SOME PRELIMINARY NOTES AND SPECULATIONS

Max S. Bell

Abstract

In this paper, Max Bell explores the nature of elementary school learning environments as they are currently conceptualized, identifies some variables appropriate to mathematics learning environments, and makes recommendations for future research endeavors. Confronting the overwhelming number of variables associated with learning environments, he offers suggestions on how to neutralize the complexity and improve reporting practices for reviews and abstracts. Bell urges that more attention be given to formulating what mathematics instruction should aim to accomplish, then focuses on the need to develop more appropriate measures of mathematics achievement and to determine the nature of mathematics learning experiences found in contemporary classrooms. Finally, Bell recommends that attention be given to possibilities for radically new mathematics learning environments within the next decade or two.

About the Author

Max S. Bell is an associate professor of education at the University of Chicago. He received his Ph.D. from the University of Michigan and previously taught at the University of Chicago Laboratory School. He is perhaps best known for his paper, "What Does 'Every Man" Really Need from School Mathematics?", which was published in Mathematics Teacher in March, 1974.

Environmental Issues in the Teaching of Mathematics in the Elementary School Grades: Some Preliminary Notes and Speculations

Max S. Bell

Introduction

My remarks here fall into two main categories. First, since the teaching of elementary school mathematics takes place in the general environment of school and the (usually) self-contained classroom, I have attempted to survey the considerable and somewhat bewildering body of research literature dealing with school and classroom environments. This survey concludes with some suggestions for sorting out the variables treated within the literature and suggestions for preventing these variables from obscuring real issues in research on mathematics learning. Second, I will outline a variety of problems with -- and make some recommendations for -- research focused on more immediate environments for mathematics learning.

The concept of "environments" was not one of my scholarly concerns until the invitation to treat this subject at the IRT Mathematics conference. My interest in curriculum, instruction, and teacher training do, of course, overlap environmental concerns, so this endeavor has been quite instructive, but I am still some distance from resolving the dilemmas that became apparent in this initial consideration of learning

environments.

I. General Elementary School Learning Environments

Sorting out issues in a field such as mathematics education for the purpose of suggesting appropriate scholarly inquiries usually requires some meta-level of discussion -- some means of conceptualizing the field, itself, and of imposing boundary conditions. Thus, we talk about such things as problem solving, the mental life of teachers, and, certainly, about environments in a variety of ways and at various conceptual levels. This approach is proper, even essential, provided that we keep track of the conceptual level at which we are speaking and provided that we don't fall into the trap of mistaking the word for the deed. We should recognize that the actual environments for school learning are what they <u>are</u> regardless of the "models," "metaphors," or whatever we might impose upon them. For example, we might decide to exclude administrative policy as an "environmental" variable for simplicity or better research design. But work with schools has shown time after time that such policy can enhance or frustrate the attempts of teachers to alter the mathematics learning environment; i.e., administrative policy <u>is</u> indeed an environmental factor whether or not we choose to attend to it.

Defining Environment

At the meta-level, the first problem is the definition of environment. If environment is to be taken as the total setting or context within which school learning takes place, then virtually everything is included: teachers (teacher training, skill, and job attitudes); socio-economic factors and their effects; buildings and classrooms; administrative policies;

curriculum materials; instruction; size of group; ratio of children to teachers (or other adults in the classroom); and so on. Even such seemingly trivial factors as amount of duplicator paper available to teachers can frustrate or enhance an attempt to alter the learning environment (as I have learned on more than one occasion). Clearly, then, without some limiting definitions, environment is everything. At the same time, however, attempts to establish more limited definitions run the risk of ignoring precisely that environmental factor that might be so powerful as to thwart attempts to manipulate the environment.

My initial approach was to define "environment" to include whatever variables are treated in the scholarly literature purporting to deal with learning environments. As I looked at articles, books, and many abstracts of articles and theses, I kept a card file of those variables which investigators called environmental. I then sorted those into some rough categories. The result was the list of approximately a dozen broad categories and one hundred variables, included here as Table 1 (p. 36-38). As I will show in more detail later, this list fails to successfully define environment, and the rest of this paper, to a very large extent, tries to deal with that failure.

The environmental rubric includes too much to be manageable. Research on environments for elementary school mathematics learning will surely be swamped by detail if it attempts to address all the general environmental variables that have been studied. Yet, as previously noted, we can hardly ignore this more general environment. The dilemmas presented by the need to

simplify environmental complexity in research on teaching mathematics and the equally pressing need to attend to it will not easily be resolved. This paper will at least outline what some of these dilemmas might be.

Boundary Conditions and "Facts of Life"

Keeping at the meta-level, I wish to draw attention to some boundary conditions that powerfully shape what we can and cannot accomplish by environmental manipulations. It is my belief that these conditions seldom become a conscious consideration in our investigations, yet their effects may explain why actual implementation of changed procedures, no matter how desirable, is always difficult and seldom achieved. These factors are environmental only in the sense that they are background "facts of life."

Order of magnitude effects. In technology, there is an axiom that an order of magnitude (factor of ten) improvement in some process can make a fundamental, qualitative change in a situation. For example, the increase in speed from travel by horses (5 mph) to automobiles (50 mph) fundamentally changed our entire society. The industrial and computer revolutions similarly brought about order of magnitude technological changes.

Perhaps order of magnitude effects also operate in such environmental variables as size of learning group and time alotted to the learning of a given subject. For instance, with respect to size of learning group, one operates entirely differently in teaching three children than in

teaching 30, and entirely differently again in teaching 300. Intuitive realization of this has made its way into common practice; few teachers, for example, attempt whole-class instruction in reading, but work with groups of five or so at a time, instead.

Scale effects. We are perpetually frustrated and puzzled in education by the finding that something that works very well in a small scale experiment proves impossible to implement on a large scale. I know how I can work with 20 teachers to change their feelings about mathematics and mathematics teaching, and I could even deal with 200; but I could not even imagine how to approach all the elementary school teachers in Chicago.

Engineers and biologists readily recognize that the transition from small to large structures can result in very different effects. Of course, they also know the laws that govern scaling: that doubling the linear dimensions of a structure, for example, changes areas by a factor of four (two-squared) and volumes by a factor of eight (two-cubed), drastically altering crucial relationships. Educators do not know the specific mechanisms that govern scale transformations, but they should surely sort out the known effects. To illustrate, it is an accepted notion by now that "the new mathematics" had essentially no effect on actual instruction in elementary schools. A plausible explanation is that we failed to offer appropriate help and direction to the million or so elementary school teachers, even though it was deemed feasible to help tens of thousands of high school teachers through summer and academic year institutes. No prejudice was involved there -- simply an escalation of scale that we could not cope with. There are often similar problems with the implementation of experimentally proved environmental changes

or pedagogic routines.

Teachers' scepticism and resistance to "academic" meddling. One can imagine fighting scale effects by some adaptation of the way in which a large number of state colleges and universities modernized agriculture through "extension services." But researchers do not have the same sort of proven dissemination procedures, nor do teachers generally see researchers as sources of help. Lortie (1975) suggests reasons for this situation:

> Teachers have a built-in resistance to change because they believe that their work environment has never permitted them to show what they can really do. Many proposals for change strike them as frivolous since they do not address issues of boundedness, psychic rewards, time scheduling, student disruption, interpersonal support, and so forth. People interested in change should take such beliefs and preferences very seriously, for they reflect first-hand experience. If disruption is chronic, subtle differences in teaching technique will hardly make the difference. If teachers become discouraged because they are short of supplies or lack backing from key adults, they will not be enthusiastic over demanding new approaches. (p. 235)

Our condescension toward practitioners and the way we talk about them and about schools in our private conversations no doubt contribute to the problem; so, too, does our general unwillingness to visit the world of practice on a sustained basis and our reluctance to "get our hands dirty" by actually applying in school classrooms our own theories and prescriptions. Whatever the reasons, teacher mistrust of researchers is a problem to which we must attend.

Misunderstanding of what school mathematics should do. Another problem in mathematics education is a pervasive narrowness and misunderstanding about the appropriate objectives of school mathematics instruction on the part of both the general public and most elementary school teachers.

This problem will be dealth with later in this paper, but suffice it to say here that this misunderstanding of objectives surely inhibits any objectives beyond teaching "basic skills" of arithmetic or any learning environments beyond exposition, drill, and practice using printed materials.

Having listed a few background difficulties with which we must cope, I will continue with an account of my attempts to come to terms with environment as a scholarly concern and some recommendations that follow from those attempts.

Environmental Concerns from Books and Monographs

As a non-specialist making a start on this subject, I consulted a series of book-length expositions about classroom life. In each case,I sought comments about environmental issues and found them in abundance.

Jackson (1968) tells us that primary school classrooms everywhere look the same and even smell the same. They are stylized, rhythmic, and cyclical environments, presided over by teachers with benignly dictatorial ambience. From a child's point of view, classrooms are characterized by three facts of life: crowds, praise/blame/evaluation, and power -- the teacher's power. The crowded environment turns the teacher into a sort of "gatekeeper" who manages the flow of discussion and activity. This situation, Jackson says, leads to a surprising amount of child time spent waiting (physically and psychologically) and as often as not, results in denial or frustration for the child. Interruptions are frequent, partly because the teacher must attempt to manage so many

souls at once and partly because of rigid and cyclic schedules, which are also an important feature of the environment.

Even while living in a crowd, Jackson states, the child must often pretend to be alone -- must ignore the other children within reach -- either to get his/her work done or to avoid breaking rules that require attention and forbid talking. The crux of his discussion is that the classroom agenda is dictated by the teacher's need to manage many children in a crowded environment. In the press of that environment, teachers live from moment to moment, he claims, relying on intuition and experience to manage situations as they arise. The teachers make few plans; they have few conceptual notions and talk about their work, if at all, in personalized and simplistic ways.

Lortie (1975) discusses the teacher's environment and finds that it is cellular, defined by the teacher and his/her students within the four walls of a classroom and isolated from all else that goes on. Teachers are the lowest functionaries in a strict hierarchy, Lortie continues. (Children are clients rather than functionaries.) They are supervised by people who don't teach. They have little or no contact concerning professional matters with others in the school. Much that affects the teachers and their teaching is not under their control. There are few rewards aside from those they may derive from working with children. Their career paths are flat; teachers go out where they came in, or else "rise" <u>out</u> of teaching into the administrative hierarchy.

For Lortie too, then, the classroom largely defines the teacher's working life. In the classroom, teachers and students are in "dual captivity" -- they have been assigned to each other with no choice on either side and must learn to tolerate or like each other for an entire year.

A good day for a teacher, Lortie says, is when the children behave and absorb the instructional material; a bad day is characterized by teacher conflicts with the children, conflicts with supervisors, and/or a feeling that the children did not learn. A new elementary school teacher comes into the classroom as a sort of Robinson Crusoe, Lortie suggests, completely on his/her own and having to formulate a comfortable -- or at least tolerable -- working environment with whatever elements are at hand. Teacher supervision is rare, and even evaluative visits are infrequent. Some teachers survive; some do not.

Dreeben (1970) examines "the nature of teaching" and encounters environmental factors at every point. He, too, finds that teachers work in isolation, with little cohesiveness with, or mutual dependence on, colleagues. Teachers are hired hands and, hence, subject to the direction of administrators, for "He who pays the piper calls the tune."

Dreeben's analysis of actual classroom life is far more sophisticated and sympathetic than Jackson's and more likely to help us define fruitful research or development efforts. The teacher's job, according to Dreeben, is still to set up classroom environments and to manage a group, yet serve individuals. The students are conscripts with allegiances to be won or compelled. The teacher lacks a "technology of teaching," so he/she cannot really predict future events; nor can he/she control with any degree of certainty what might happen. Certain types of errors are difficult to retrieve. Dreeben emphasizes the central importance of classroom control. He observes that, with no technology of teaching, the teacher must rely on charisma, personality, and other

elusive and personal factors, or perhaps simply on dominance, power and discipline. But Dreeben does not consider all this inevitable. He argues persuasively for studies beginning with the actual classroom environment. He sees such studies as starting points for developing technologies of teaching that would help make teaching processes more deliberate and predictable. The technologies, he says, could bring about substantial change in traditional classroom environment.

Thelen (1960) finds existing classroom environments for the most part unacceptable. He maintains that various kinds of individual and group inquiry are the keys to learning and that environments very different from that of the contemporary school classroom are needed to promote these modes of inquiry. Only for "skills development" (and Thelen does stress the importance of learning certain skills) should learning environments resemble those in existance. For our purposes, it should be noted that the application of Thelen's recommendations is dependent upon manipulating learning environments, including efforts to achieve happier and more compatible grouping of students with teachers.

Jackson, Lortie, Dreeben, and Thelen essentially agree that management, control, and discipline play a central role in classroom environments. Kounin's book (1970) is relevant to this discussion for two reasons. First, it provides an excellent argument that forms of management of classroom groups (not merely individuals) are strongly related to the involvement of youngsters in their schoolwork, the incidence of "deviant behavior," and the ways in which children judge and react to their teachers.

In addition, it is instructive to trace the process of Kounin's investigations. From classroom observation, he thought he noticed a "ripple effect" from discipline of individuals. In setting up studies of this perceived effect, he discovered that extensive classroom observation failed to provide the detail concerning complex interactions that would be needed for verification; thus, he moved to videotaped sessions that could be examined at leisure. Along the way, he found that the supposed "ripple effect" could not be confirmed, but he continued the investigation to examine other possible effects. Those further investigations produced some interesting concepts which he discusses in some detail.

Kounin proposes that good or poor management is defined by two central dependent variables: "work involvement" and "freedom from deviancy." His videotapes allowed him to quantify those variables and a number of other independent variables, some of which are listed in Part C of Table 1. Identification of such variables is useful. Furthermore, the investigative process used by Kounin serves as something of a model for investigation of what goes on in classrooms, though I would prefer to see more input from the practicing teachers as actual collaborators, and not merely as subjects.

In their review of conceptions of classroom instruction, Barr and Dreeben (1977) contrast production function research (analysis of inputs and outputs) with several other methods of research on classroom processes.*

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* For this discussion, I worked from a manuscript which Barr and Dreeban gave me permission to use, even though it was not yet ready for publication or attribution. A revised version will appear in the Review of Research in Education: Vol. V, in press.

With regard to production function research, they note that the technology by which inputs are turned into outputs is largely unknown. Without a conceptualization of that technology, they say, "the study of school productivity amounts to little more than the contemplation of black boxes and the spinning of ever greater fantasies about what happens inside them." The obvious solution is to look closely at classrooms, themselves -- at the insides of the black boxes. The black boxes, of course, contain many environmental variables.

Barr and Dreeben discuss six prototypes for research into actual classroom process: Flanders on social interaction; Bellack on the language of the classroom; Kounin on classroom management; Bennett on teaching styles; and two manifestations of concern with individual differences -- Dahloff on frame theory and Bloom on mastery learning. Each of these involves environmental manipulations of one kind or another. In summary, Barr and Dreeben note that two characteristics limiting the usefulness of classroom studies are the "tendency to examine only a portion of the events occurring within classrooms and the tendency to ignore the contextual properties of classrooms." But to link classroom studies to production function research, "we need to know what classrooms are like, how they work, and what situations arise in them that entail the allocation of resources." They note that classroom studies to date have not sufficiently supplied that understanding.

The Barr and Dreeben critique is sharp and just. It helps create an awareness of the weaknesses and the possibilities of research attempts to characterize what goes on in classrooms regarding both environments and instruction.

Environmental Concerns in Journal, Dissertation, and Review Literature

There are numerous articles and dissertations indexed under the rubric of "classroom environments" in Current Index to Journals in Education (CIJE) and Psychological Abstracts (PA) -- about 50 per year in CIJE and more than that in PA, with very little overlap in the titles. In addition, there are the "see alsos," such as "school environments" and "home environments" -- categories that a complete review would have to assess.

Quite a few of the CIJE and PA classroom environments abstracts give information or suggest ideas about mathematics learning. However, of the nearly 200 articles indexed under "elementary school mathematics" in CIJE alone, only 10 or so seem to relate to environmental concerns. My net conclusion is that, to find extensive information on environments for learning mathematics, one should look outside the literature of mathematics education, as such.

I will not claim to have examined this vast literature in great detail, but I was able to make some interesting discoveries. For example, much of the work reported lately in theses and articles deals in one way or another with "open" versus "conventional" classroom environments. Yet, recent annual compilations of research on mathematics education, such as Suydam-and-Weaver's in the Journal for Research in Mathematics Education, include very few studies of the effects of "open" environments on the learning of mathematics.

The sheer number of environmental variables treated in books or research reports about classroom life is almost overwhelming; even my listing in Table 1 is not complete. Clearly, I was naive in hoping to neatly summarize the research on the general classroom environments in which mathematics instruction is embedded. Having made the attempt, however, I

cannot put it aside without trying to suggest ways to organize and conceptualize that large collection of variables.

Two Suggestions for Conceptualizing the Collection of Environmental Variables

One way to deal with the complicated matter of general environments for learning is to formulate a kind of vector space of environmental variables. I choose the vector space analogy mainly because it suggests tools that might be helpful in thinking further about a problem like this. For example, for each variable in a vector space, a measure and a range of values must be defined; this task aids in the understanding of the variables. In a vector space, scalar multiplication can serve to assign weights to variables, and linear combinations of variables allow for the expression of certain kinds of interactive effects. One can think about what might be a basis for a given environmental vector space, and about alternate bases, reduction of basis, etc. Also, since factor analysis is closely linked to vector spaces, starting with some vector space analogy potentially links the inquiry to a familiar statistical tool.

As I worked at trying to represent learning environments as a many-variable vector space, however, I found that such a variable-by-variable listing still fails to characterize some fairly obvious and essential background conditions that shape learning environments, generally, and mathematics learning environments, specifically. Nevertheless, the vector space analogy seems a useful image that may be worth refining further.

Another conceptualization of mathematics learning environments is one of clusters or systems of variables, each cluster distinct, yet with considerable overlap. The appropriate diagram for showing these systems and interactions among them can not be like an organization chart with some sort of hierarchy implied, although it is clear that some systems

have more powerful effects than others. Nor can the appropriate diagram resemble a flow chart -- the links and relationships are certainly not that clear. A diagram showing clouds of categories with interaction arrows might do, except that it would fail to show the overlap among systems. Perhaps some form of a Venn diagram would serve best, though its specific intersections would be difficult to sort out and illustrate. It is the overlap and seepage across boundaries that makes such pictures potentially more useful than a vector space picture.

Whatever conceptual model is attempted, it is important to identify the focus of attention in collecting environmental variables. Otherwise, the researcher is likely to be completely swamped by the complexity of "environments" for learning. Also, it is essential that the environmental effects not be construed as mutually exclusive, regardless of the conceptual model selected. Strict ordering of influence of variables is probably not possible, although it is clear that variables are not equal in weight and some have powerful determining effects. For example, the Chicago schools' perpetual money bind has, on several occasions, led to sudden and apparently arbitrary fundamental alterations which have wiped out experiments or development efforts in specific schools. As this case indicates, certain socio-political influences can very quickly undermine specific efforts to alter a given classroom environment.

Conclusions and Recommendations

The complexity and number of variables linked to classroom environments have become very evident to me. It is obvious, too, that since most elementary school mathematics education takes place in self-con-

tained classrooms, whatever affects those classroom environments is bound to affect mathematics learning environments; the two are essentially the same thing.

Yet, it is necessary to neutralize the complexity of general environments if we are to obtain any specific information about mathematics learning environments; hence, my first recommendation:

> In conducting research on environments for elementary school mathematics learning, methods must be found to neutralize the confounding effects of more general learning environment.

Following are three suggestions as to how that neutralization might be accomplished (there are no doubt other possibilities):

1. Conduct at least initial studies only under environmentally favorable conditions. Initial research on mathematics learning in self-contained classrooms could be restricted to situations where: (1) the teacher has the general environmental situation well in hand, and (2)the teachers are considered"excellent" by a consensus of those who know teaching. We would undoubtedly find that having the general situation in hand would not guarantee a good mathematics learning experience. That is, the variance in mathematics learning environments (and in instruction) would still be considerable even in classes already screened for generally favorable ambience.

2. Set up mathematics learning groups that are independent of self-contained classrooms; these could be with teachers expert in teaching mathemtaics and proficient in managing classrooms, motivating students, etc. Alternatively, there could be special environments set up in which classroom management tasks would be of less central importance than usual or where management would take different forms than usual.

3. Set up carefully randomized, relatively large scale "social experiments" to evenly distribute general environmental effects as background "noise." Rieken and Boruch (1974) provide a good discussion of those possibilities.

These recommendations for mathematics education research emanated

from a look at rather general issues. The following is a discussion of the specific issue in the elementary school mathematics learning experience.

II. INQUIRIES INTO ELEMENTARY SCHOOL MATHEMATICS LEARNING ENVIRONMENTS

Rather than summarize and review recent research on the elementary school mathematics learning environment, I will devote this section to a discussion of the sometimes disturbing questions which are raised by research in this area. A number of fundamental issues concerning future research initiatives will also be addressed.

The results from environmental research in mathematics education conducted to date indicate that:

1. Except for time-on-task, environmental manipulations seldom yield markedly better results than other manipulations or no manipulation at all. Zero, negative, or mixed results are the rule.

2. Children from "more favorable" school environments do better on present standardized achievement tests; the same applies for children from more prosperous home and social environments. I don't believe there are necessarily cause-and-effect links here; in fact, I think most of these associations can be accounted for by such variables as effective time-on-task and richness of the early childhood environment.

3. Children tend to learn best what is emphasized most.

4. Teachers don't know much about mathematics.

5. Teachers like to teach arithmetic (This is not really inconsistent with the previous statement, given the obvious distinctions between arithmetic and mathematics).

That may seem like a scant yield from a lot of effort but, in addition, the large and growing body of carefully-executed research results suggests a number of interesting hypotheses and a variety of important environmental variables. To verify that, an interested reader might consult the annual compilations of research by M. Suydam and F. Weaver in the

Journal of Research in Mathematics Education, skim the section on learning environments in Callahan and Glennon (1975), or spend time with "classroom environments" in Psychological Abstracts.

Difficulties in Research on Mathematics Learning Environments.

As just noted, many fruitful hypotheses can be derived from available reports of research on learning environments. At the same time, an examination of that research raises troublesome questions about the research itself and the reporting of it, as well as fundamental questions about the present state of elementary school mathematics learning and teaching. Consideration of both the methodological and status questions would seem important in planning a program of research in mathematics education. Below is a list of such questions; in the sections that follow, each will be discussed.

1. Are existing reviews and abstracts of mathematics education research adequate and helpful?
2. What is the present definition of "elementary school mathematics" and is it acceptable as the basis for research in early mathematics education?
3. What measures currently define "mathematics achievement" for elementary school children and do they provide an adequate basis for research?
4. Does a viable elementary school mathematics learning experience exist for most children?
5. To what extent can future research on environments for elementary school mathematics learning build on firm knowledge from past research?
6. Are there possibilities for radically new mathematics learning environments within the next decade or so?

Question 1: Are existing reviews and abstracts of mathematics education research adequate and helpful?

Mathematics education research has made significant progress in

the past decade or so in a number of ways; but there are not yet fully satisfactory mechanisms for abstracts and reviews of research studies in mathematics education. Even where they exist, it is often unsafe to rely on the conclusions stated by abstracts or summary reviews. For example, summary reviews of a body of research focused on specific issues necessarily attend only to main findings. But even when it would be important to do so, many such reviews fail to characterize such details from the separate studies as the settings for the research, ages of subjects, time spans over which the research extended, and other things that, if attended to, might fundamentally alter the the conclusions given by the review.

Abstracts of single research studies are often marked by similar deficiencies. For example, they frequently provide insufficient information about the time span over which sometimes substantial environmental alterations were attempted--surely a crucial matter. They often omit important information about number of research subjects or their ages. Also, crucial details about treatments are frequently left out.

Variables are often mislabeled in abstracts. To take a common and notorious example, what the abstracts call "mathematics achievement" without further detail is usually completely defined by tests of calculation achievement, which is surely a different matter than mathematics achievement.

Abstracts and reviews seldom provide information needed to draw distinctions between statistical significance and significance. A review or an abstract might indicate that a certain manipulation of learning environments produced differences "significant at the .05 level" but not say, for example, that the statistically significant

result was produced by average scores of 34.3 percent vs. 30.6 percent. Of course, that additional information might raise obvious questions about whether such a difference in scores indicates substantial difference in competence or why the average scores were so low.

The problems just outlined concerning abstracts and reviews lead to my next recommendation:

> Improved mechanisms should be found to assure skillful and critical abstracting of reports dealing with research on teaching elementary school mathematics and to make available periodic reviews of related research.

The abstracts should in every appropriate case include the following information (or note its absence from the original research report): age of subjects; number of subjects; length of time during which an intervention or treatment was carried out; criteria used to define "achievement" or "attitude" or other variables under study; total scores possible; enough information on results to assess practical effects as well as statistical significance; unusual characteristics of research setting or participants; and, where possible, some comment on the generalizability of results.

Question 2: What is the present definition of "elementary school mathematics" and is it acceptable as the basis for research in early mathematics education?

It has long been my impression that for most children in grades K-6 "the calculation curriculum" and "the mathematics curriculum" are essentially synonymous; that is, concern with teaching calculation dominates the elementary school mathematics learning experience. I

have often found that even when topics like measure, geometry, and probability are in the textbooks, they are either skipped or used only to provide another context for calculation. In order to check on those impressions, I have recently tried to assess the K-6 mathematics curriculum by analyzing the content and sequencing of topics in standard textbooks series for grades K-8, visiting classrooms, studying results from surveys of teacher practices and opinions, and noting the results of the National Assessment of Educational Progress (NAEP) mathematics assessment. From those exercises, I am even more persuaded that elementary school mathematics is, for most children, largely defined by the teaching of calculation. In that calculation curriculum, very little is done through the first three grades. The pace picks up a bit in grade 4. In grades 5 and 6, however, a lot of complicated and highly symbolic material is introduced, with almost no advance student preparation. From first grade on, the calculation curriculum emphasizes symbols; there is little concern with playfulness about patterns and structures, uses of mathematics, or concrete embodiments of concepts. Drill and practice with printed materials dominates. Time allocations in the early grades appear to be minimal (as I will discuss below). Achievement testing is often a prominent part of the psychic environment, for teacher effectiveness is increasingly judged by children's performance on "basic skills."

Moreover, even in its main aim, the calculation curriculum achieves only limited results. It is true that most students do learn to add, subtract, multiply, and even divide with whole numbers though many don't do so well with fractions or decimals (NAEP, 1975a). But these same students often can't use those operations. That is, in practical situations, many children cannot decide <u>what</u> to add or <u>whether</u> to add

(maybe they should multiply?), or, having added, they have no idea whether or not their answers make sense in the situation at hand (NAEP, 1975b). Surely that is a non-trivial failure of present-day instruction.

I am aware that these remarks poach on the curriculum preserve covered elsewhere in this conference, but the question of what mathematics children are asked to learn is so fundamental that it can hardly be ignored in considering environments for learning. Hence, let me state as strongly as possible that the restriction of elementary school mathematics to basic arithmetic calculation is counter-productive and unhealthy. Two alternatives to that narrow definition are included here as appendices 1 and 2 (NCSM, 1977; Bell, 1974).

This discussion leads to my next recommendation:

In doing research on elementary school mathematics teaching, we should not merely accept as given and unchangeable the narrow objectives and aspirations that presently define that enterprise. To do so makes that research essentially trivial.

Question 3: What measures currently define "mathematics achievement" for elementary school children and do they provide an adequate basis for research?

In virtually all environmental research that refers to "mathematics achievement" as a variable, what is actually being discussed is "computation achievement," and the measures used to assess even that frequently have serious flaws. Those measures almost always consist of scores on one or another standardized test, such as the Iowa Test of Basic Skills (ITBS) or the Metropolitan Achievement Test (MAT). Raw scores on such tests reveal little detail about what children can or cannot

do, even with respect to computation--they are pretty much like a doctor who asks how much something hurts rather than where. For example, for the eighth grade version of the ITBS, 24 correct answers out of 48 puts a student at "grade level," but of course the number of ways to get 24 of 48 answers correct is enormous. I suggest that such scores, by themselves, have essentially no relevance in measuring the efficacy of manipulations of mathematics learning environments.

If we are to conduct studies of elementary school mathematics learning, we must first decide what learning skills should be stressed, where "skills" has a broader meaning than it is often given. (For that, I have already suggested the NCSM list as a rough guide.) We must also find ways to measure the extent to which the emphasized skills are, in fact, learned and for that there are at present few easily usable instruments. Hence, this recommendation:

> A first order of business for research efforts focused on elementary school mathematics learning should be development of good instruments and procedures that will give reliable, valid, and detailed information about what mathematics children actually learn in their school experiences, including learnings other than computation.

Question 4: Does a viable elementary school mathematics learning experience exist for most children?

For the primary school grades, at least, there are reasons to believe that a viable mathematics learning experience may not exist for many youngsters -- perhaps even for a majority. Some of these reasons are as follows:

Overemphasis on calculation. Mathematics learning as defined by NCSM (1977) or Bell (1974) does not exist in the great majority of classrooms; most schools focus merely on the learning of calculation (arithmetic). But, for present purposes, let us define the "mathematics learning experience" as an "arithmetic learning experience" and check further on whether even that is viable.

Time spent teaching mathematics. Although reliable and detailed information is lacking, there is some evidence that little time is devoted to teaching arithmetic in elementary grades. The Chicago Board of Education, for example, gives its teachers detailed time specifications (see Appendix 4) which call for 50 to 100 minutes per week to be spent on mathematics in first grade, increasing to 150 to 225 minutes per week by sixth grade. Assuming teachers adhere to such directives and taking the middle values, that yields a specification of 15 minutes a day in first grade, rising to 38 minutes a day in sixth grade. In a small study in a Chicago suburban school system, we found that about 25 minutes per day were spent on mathematics in third and fourth grade. In several questionnaire surveys of teachers, conducted for the National Advisory Committee for Mathematics Education (NACOME) (1975), Research for Better Schools (RBS) (Graeber, 1977), and myself (Bell, 1976), respondents have reported spending an average of 40 minutes per day in mathematics instruction (less in lower grades, more in upper grades). Follow-up interviews in the RBS study indicate that we can discount that figure by perhaps 20 percent, even for these self-selected respondents.

A study in which observers followed 477 teachers around for a full day (Conant, 1974) revealed that those teachers spent an average of only 100 minutes (about 30%) of a 5.5-hour school day on activities

related to academic learning and instruction. Of that 100 minutes, 75 were devoted to language arts, 18 to number work, and no more than one or two minutes each to any other curriculum area. Another report (Barley, 1975) indicates that elementary school youngsters spend "3% of their waking lives" on mathematics (5% on reading), including work outside of school. Assuming the report refers to school days, at 14 hours awake, this would amount to about 22 minutes per day.

In sum, a variety of limited studies seem to suggest that the allocations of 15 to 38 minutes per day specified by the Chicago Public Schools are representative of the general time situation. Such scant time allocations are surely inadequate, to say the least, even if very efficiently used.

Inefficient use of mathematics time. We can probably assume with Jackson (1968) that much of a child's working time is, in fact, waiting time. I estimate on the basis of talks with teachers that, for given children, at least 25% of each lesson period is lost in this way. If so, the Chicago specifications of 15 to 38 minutes per day shrink to 12 to 30 minutes of actual involvement; and those figures might represent maximums. (It should be noted that such severe time limitations virtually exclude manipulative materials, projects, small group investigations, etc. from the learning environment.)

Threshold for usable mathematics learning. With scant time devoted to arithmetic, inattention to task, frequent interruption

of concentration, or some combination of these factors, it might be that much of the arithmetic learning experience never reaches the essential threshold that would characterize durable and usable learning. Perhaps, day after day, many things are not quite learned, or at best, only partially learned. If essential thresholds are not reached, the effective existence of an arithmetic learning experience is certainly diluted.

Students' dislike of arithmetic. Surveys show that although the majority of children profess to like arithmetic, a sizable number do not. For them, the learning experience is often a negative one, and the sum of positive and negative learnings may be near zero, although I would not insist on this somewhat fanciful speculation.

Limited content. The strongest indication that arithmetic teaching barely exists is to observe how little content is covered during six to nine years of elementary schooling. This content includes mainly the four standard operations with whole numbers, fractions, and decimals, and that makes a pretty thin spread of content, even if completely learned.

I do not claim that my pessimistic speculations are all correct. But I do think there are real and serious questions that call for status studies in at least some of these areas. Certainly, considering what most people can actually do with mathematics, it is clear that the mathematics learning experience is seriously deficient, even if it does exist.

Question 5: To what extent can future research on environments for elementary school mathematics learning build on firm knowledge from past research?

Many interesting hypotheses can be drawn from past research results and from the common wisdom and intuition of thoughtful people. Both of these sources provide fruitful starting points for clinical and empiricial research studies. On the other hand, the following are some reasons to believe that past research doesn't give us much to go on in planning future research on environments.

1. Although there are a number of special existence proofs showing that this or that fine result can be obtained with children in special environments or by special instructional routines, very few of these special procedures or environments have either made their way into common classroom practice or been subjected to follow-up research studies.

2. There is relatively little research in mathematics education focused on environmental variables. Most information about such variables comes from more general studies of environments. The bulk of mathematics education research is focused on curriculum "treatments," and what remains is mostly concerned with methodological or instruction variables.

3. Although research involving the manipulation of general environments (e.g., open classroom, non-graded situations) sometimes includes incidental information on "mathematics achievement" (usually measured by standardized tests), that research generally has mixed or null results.

4. Even when definite results are reported in research on manipulation of environments, those results are often suspect, for reasons already discussed: (a) "achievement" is defined by narrow and misguided objectives and measured inadequately, and (b) many studies (especially for doctoral theses) are carried out over such short time periods that it is unlikely that durable effects could have been achieved.

With such basic shortcomings in previous research, I believe that environmental researchers should proceed almost as if they were starting from scratch. They should also see to the compilation of excellent reviews and summaries of those reliable research results that do exist. Perhaps those reviews will indicate that I am too pessimistic about past research.

Question 6: Are there possibilities for radically new mathematics learning environments within the next decade or so?

The availability of small electronic calculators in the United States will soon approximate that of TV sets, refrigerators, and automobiles. Hence, many of the hard-won calculation skills from years of schooling may soon seem irrelevant to nearly everyone outside of school. (What actually remains relevant in school learning in spite of this seeming irrelevance is complicated and can't be discussed here.) Furthermore, tens of millions of people outside of school will have access to sophisticated hand-held computer power, to automatically programmed routines for many ordinary transactions, and to powers of interactive information manipulation and display that are rare even in full-scale computer centers today. Moreover, there appear to be no technological barriers to giving homes (or schools) access via cable TV or telephone lines to enormous banks of information, visual displays, etc. Anything which exists in retrievable form, including many well-developed but often inaccessible films illustrating mathematical concepts and applications, could in theory be included in such data banks. The relentless

quest of the high technology companies for cost reduction, miniaturization, and reliability will continue to lead to order-of-magnitude changes in cost, capacity, and size, and, hence, to increasing availability.

One should, of course, retain some skepticism about whether such visions of better living (and learning) through microelectronics will actually materialize. We must also remind ourselves that, at least over a short run of 10 to 20 years, schools have an extraordinary capacity to go their own way with little change no matter what happens in the outside world. Yet, the prospect of rather startling technological developments is great enough that the IRT should consider sponsoring attempts to examine the possibilities for substantial and fundamental changes in the child's learning environment.

Some Specific Variables for Research on Mathematics Learning Environments

At the conclusion of Part I, I recommended several ways of neutralizing the complexity and confounding effects of general environments in order to pinpoint variables related specifically to mathematics learning. Here is a rough list of what some of those specific variables might be:

1. Effective time-on-task, as some combination of:
   a. Actual time allocated to and spent on mathematics instruction;
   b. Other time spent on mathematics, e.g., optional time or mathematical subject matter appearing in other subjects;
   c. Number of students per instructor, which relates to help available, waiting time, etc;
   d. Intensity of involvement of the child with the task.
2. Materials available in the learning environment:
   a. Ranging from printed materials to manipulative materials;

b. For print materials, ranging from expository to exercises;

c. For manipulative materials, ranging in structure from free play to highly rule-bound.

3. Learning group size, ranging from individual work to full class.

4. Learning group character, including:

a. Individual work;

b. Group work, ranging from peer interactive to adult-dominated;

c. Prescribed learning mode, ranging from inquiry-oriented to completely receptive.

5. Learning group homogeneity (which appears to be quite important to teachers) with respect to:

a. Previous exposure to material at hand;

b. Previous success or "achievement";

c. Age;

d. Out-of-school background;

e. "Intelligence" or quickness in absorbing the type of material at hand, whatever it may be.

6. "Disruptive" influences within learning group, as evidenced by:

a. Responsiveness, ranging from all children willing to conform to behavior demanded by the task at hand to none willing;

b. Variations in responsiveness from task to task, e.g., children willing to go through drill sheets forever might soon become restive with manipulative materials.

7. Teacher expertness with respect to:

a. Mathematics generally, suggesting ability to respond to non-standard child responses;

b. Particular objectives agreed upon (a teacher expert in teaching of computation may feel uncomfortable with measure, problem solving, approximation, geometry, probability, etc.)

c. Setting up and managing the mode of instruction used -- "individualized," small group, expository to large group, "discovery," etc.;

d. Diagnostic and prescriptive routines.

8. Stated content objectives, ranging from computation only to something like the NCSM specifications.

9. General ambience, e.g., encouragement, warmth, openness, expectations conveyed, etc. (This, obviously, needs more careful specification.)

Some Suggested Research Initiatives

It would be futile to make a long list of aspects of mathematics learning environments that "must" be illuminated by research. Instead, let us settle on a short list that reflects some sense of our limitations and our priorities. Here is my suggestion for such a short list of research initiatives:

Development of better measures of achievement (not merely computation achievement) and attitudes. To develop better measures, of course, it is essential that agreement be reached on a set of goals or objectives for the school experience. I have tried to provide some guidance on that.

Carrying out status studies and surveys. These would be designed to map out common practice and uncover notably successful exceptions to common practice. Status studies would also be directed at critical reviews to determine what we "know" from previous research and to identify the interesting hypotheses and possibilities not yet fully tested. Other status studies would aim to identify potentially helpful curriculum and instruction routines that exist but are unused.

Intervention in classrooms or classroom studies. One such type of study would simply involve visiting a number of classrooms on a regular basis and observing how things are done, intensity of student involvement, what barriers there appear to be to improved mathematics education, what are the time allocations, etc. Another kind of study would involve implementing Lortie's (1975) suggestion that we work with teachers and supply them with what they say they need, perhaps also supplying them with guidance on how mathematics education itself needs to change. A third type of study would have "experts" in mathematics education team up with "excellent" teachers, work with them long enough to gain a feel for the realities of elementary school classroom life, and then stay on to see what changes could be made in the normal mathematics education experience of youngsters. In such a study, we should carefully note realistic restraints and aim to make our "remedies" workable without us.

A good way to approach this research would be to follow up on the national sample of schools used by the National Assessment of

Educational Progress. There is already quite a bit known about these schools, and anomalies such as markedly superior achievement could easily be detected from work already done. Moreover, on-the-scene classroom observations and studies would greatly extend the information already in NAEP files.

In conducting these studies, we should not hesitate to seek the assistance of specialists in survey and opinion studies, such as the National Opinion Research Center or Gallup.

Conclusion

This attempt to sort out elementary school mathematics learning environments proved to be a stimulating, but frustrating exercise. In the course of it, I discovered many obstacles to, and problems with, effective mathematics learning and mathematics learning research. I have attempted to provide recommendations to surmount these obstacles and overcome those problems. Perhaps the best way to conclude would be to call attention once more to those recommendations; in summary, I have proposed that we should:

(1) Find ways to neutralize or finesse the complications in general learning environments to keep from swamping specific research efforts (p. 16)

(2) Attend to better abstracting and review of existing research results (p. 20)

(3) Formulate a better concept of what school mathematics instruction should aim to accomplish (p. 23)

(4) Develop more adequate means of measuring what we aim to accomplish (p. 24)

(5) Undertake status studies to ascertain to what extent an adequate mathematical learning experience exists, especially in the primary school grades (p. 27)

(6) Conduct exploratory studies to prepare for the possibility that radically new learning environments will be feasible within the decade (p. 30).

To this list could be added the recommendations implicit in the priorities I have suggested for research in mathematics learning environments (p. 32-33) and, of course, a demand for attention to the variables identified on pages 30-32

Follow-up on these recommendations will involve a great deal of effort, but much of it can be incorporated into other work on curriculum, instruction, teacher training, and studies of teaching proposed at this conference. I believe we can settle on a set of manageable tasks upon which IRT can make substantial headway.

Appendix 1

A Preliminary Attempt to List Variables in the

General Learning Environment

A. Variables in the working environments of teachers that are often not under their direct control (Note: To the extent that these shape teacher actions, comfort with with his working situation, etc., these probably effect classroom environments to a non-trivial extent.)

1.	Supervision	(age, sex, and compatibility of supervisor; purpose of supervision, e.g. evaluation vs. help; frequency; obtrusiveness; colleagual or administrative; expertness of supervisor; guidelines; specificity of expectations; etc.)
2.	Colleagues	(quality; compatibility; colleaguality; interdependence; common time for professional contact; etc.)
3.	Assigned space	(size; furniture, e.g., fixed or moveable; lighting; temperature; ventilation; physical condition; black-boards; other display space; storage space, general ambience; etc.)
4.	General school atmosphere	(freedom of movement of children; orderly/ chaotic; noise; PA systems and use of same; interuptions; etc.)
5.	Scheduling	(imposed time constraints, imposed sequences of events; etc.)
6.	Administrative policy	(flexibility; teacher participation; arbitrariness; how bureaucratic; focus, e.g. on children, on support of teacher, on ease of administration, on enforcing rules; etc.)
7.	"Authority of lay opinion"	(approximately degree and type of parent involvement)
8.	Training opportunities	(within school inducements to take outside courses; chances to observe other teachers; released time or not; compulsory or not; etc.)
9.	Administrative support	(In "discipline,"; with respect to parents; in getting materials; in special arrangements; etc.)

10.	Occupational status	(job security; autonomy; salary and fringe benefits; etc.)
11.	Imposed "philosophy" and policy	(For example: self contained classrooms or specialists; "this school <u>is</u> an open school"; "accountability"; team teaching; "continuous progress"; "individualized education"; lesson plans required; non-teaching responsibilities assigned; etc.)

B. Variables in the environment that are <u>within</u> a given teacher:

Expertness in various subject matters; general intelligence; special talents and gifts; breadth of interests; philosophy of learning and teaching; comfort with adult relationships; comfort with children; flexibility, and importance attached to that; warmth, and importance attached to that; intensity; mannerisms; interaction needs--with adults or with children; general psychological health; voice projection; voice quality; "fairness" (as perceived by pupils); preferences; prejudices; physical health; stamina; self pacing; listening ability; and so on.

C. Variables in the environment likely to be under the control of a teacher

(Note: The foregoing makes clear that much of the classroom environment is not directly or consciously controlled by the teacher. The craft of teaching may consist in the manipulation of what is left to teacher control. Here is a partial and somewhat random listing.)

1.	Classroom appearance	(Physical arrangements; bulletin board and other displays; general appearance and decoration of classroom; etc.)
2.	Grouping	(Groupings of students for various purposes; freedom of movement; freedom of expression; solitariness or groupiness of various learning experiences; etc.)
3.	Materials	(Materials available (controlled within limits by the teacher); materials actually used; access of children to materials; concreteness of materials; extent of use of resources and settings outside the classroom; etc.)
4.	Management	(The Kounin classroom management variables: "withitness" (including accuracy of target and timing in reprimands and praise); overlapping; smoothness; momentum; group alerting; accountability (extent to which teacher makes students responsible); format; satiation/variety.)

5.	Interactions	(Playfullness; warmth; humor; ease of interaction; formality;informality; flexibility; arbitrariness; praise/blame/reward; consideration/bullying; even-handedness; etc.)
6.	Scheduling	(Distribution of time among subjects; variety of subjects; relative emphasis)
7.	Expectations	(encouragement; respect for children; expectations conveyed.)

D. Classroom environments as perhaps seen by children.

(Note: This is shaped in obvious ways by A-C above and E below. In addition here are some factors from the literature or from the"common wisdom," listed randomly.)

Stable physical environments; stable rituals, schedules, routines; stable behavior rules; crowds, waiting (physical and psychological); denial; delays; interruption; interaction (child-child, child-group, child-teacher); extent of freedom of movement; extent of autonomy; variations in learning group size; congruence of personality with teacher; congruence of personality with teacher; conflicts with other children, with school adults; need to be alone in crowd; intensity of experience; anxiety; orderliness or chaos in classroom and in school; comfort of the physical environment; congruence of past learnings to present expectations; group norms; range and level of ability of classmates; willingness to be in school; student disruptions; school quality; richness of visual environment; noise; variety of experiences offered; modality of materials (printed to concrete) and congruence with learning style or needs; number of adults in classroom; opportunities for solitude, reflection; diagnoses; evaluation, testing; control over or controlled by events

E. Environmental variables from the community in which the school is imbedded. They are legion, including:

community support of schools; teacher/client or teacher/parent social congruity; stability of pupil group; community SES factors; community tensions; parent education; family characteristics; ethnicity, language variations, etc.

Appendix 2

A List of Basic Skills in Mathematics for Everyman

From: "What Does 'Everyman' Really Need from School Mathematics?" M. S. Bell. The Mathematics Teacher, March, 1974, p. 199.

1. The main *uses* of numbers (without calculation)	1.1 Counting 1.2 Measuring 1.3 Coordinate systems 1.4 Ordering 1.5 Indexing 1.6 Identification numbers, codes 1.7 Ratios
2. Efficient and informed use of computation algorithms	2.1 Intelligent use of mechanical aids to calculation
3. Relations such as equal, equivalent, less or greater, congruent, similar, parallel, perpendicular, subset, etc.	3.1 Existence of many equivalence classes 3.2 Flexible selection and use of appropriate elements from equivalence classes (e.g., for fractions, equations, etc.)
4. Fundamental measure concepts	4.1 "Measure functions" as a unifying concept 4.2 Practical problems: role of "unit"; instrumentation; closeness of approximation 4.3 Pervasive role of measures in applications 4.4 Derived measures via formulas and other mathematical models
5. Confident, ready, and informed use of estimates and approximations	5.1 "Number sense" 5.2 Rapid and accurate calculation with one and two digit numbers 5.3 Appropriate calculation via positive and negative powers of ten 5.4 Order of magnitude 5.5 Guess and verify procedures; recursive processes 5.6 "Measure sense" 5.7 Use of appropriate ratios 5.8 Rules of Thumb; rough conversions (e.g., "a pint is a pound"); standard modules 5.9 Awareness of reasonable cost or amount in a variety of situations
6. Links between "the world of mathematics" and "the world of reality"	6.1 Via building and using "mathematical models" 6.2 Via concrete "embodiments" of mathematical ideas
7. Uses of variables	7.1 In formulas 7.2 In equations 7.3 In functions 7.4 For stating axioms and properties 7.5 As parameters
8. Correspondences, mappings, functions, transformations	8.1 Inputs, outputs, appropriateness of these for a given situation 8.2 Composition ("If this happens, and then that, what is combined result?") 8.3 Use of representational and coordinate graphs
9. Basic logic	9.1 "Starting points": agreements (axioms) and primitive (undefined) words 9.2 Consequences of altering axioms (rules) 9.3 Arbitrariness of definitions; need for precise definition 9.4 Quantifiers (all, some, there exists, etc.) 9.5 Putting together a logical argument
10. "Chance," fundamental probability ideas, descriptive statistics	10.1 Prediction of mass behavior vs. unpredictability of single events 10.2 Representative sampling from populations 10.3 Description via arithmetic average, median, standard deviation
11. Geometric relations in plane and space	11.1 Visual sensitivity 11.2 Standard geometry properties and their application 11.3 Projections from three to two dimensions
12. Interpretation of informational graphs	12.1 Appropriate scales, labels, etc. 12.2 Alertness to misleading messages
13. Computer uses	13.1 Capabilities and limitations 13.2 "Flow chart" organization of problems for communication with computer

Fig. 2. A short and tentative list of what is "really" wanted as a minimum residue for everyman from the school mathematics experience

Appendix 3. A List of Basic Skills from the National Council of Supervisors of Mathematics

TEN BASIC SKILL AREAS

Problem Solving

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.

Applying Mathematics to Everyday Situations

The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics, and interpret the results in light of the initial situation.

Alertness to the Reasonableness of Results

Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

Estimation and Approximation

Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.

Appropriate Computational Skills

Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single-digit number facts is essential and mental arithmetic is a valuable skill. Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.

Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

Geometry

Students should learn the geometric concepts they will need to function effectively in the 3-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular. They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills. They also must be able to recognize similarities and differences among objects.

Measurement

As a minimum skill, students should be able to measure distance, weight, time, capacity, and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

Reading, Interpreting, and Constructing Tables, Charts, and Graphs

Students should know how to read and draw conclusions from simple tables, maps, charts, and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts, and graphs.

Using Mathematics to Predict

Students should learn how elementary notions of probability are used to determine the likelihood of future events. They should learn to identify situations where immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make predictions such as election forecasts.

Computer Literacy

It is important for all citizens to understand what computers can and cannot do. Students should be aware of the many uses of computers in society, such as their use in teaching/learning, financial transactions, and information storage and retrieval. The "mystique" surrounding computers is disturbing and can put persons with no understanding of computers at a disadvantage. The increasing use of computers by government, industry, and business demands an awareness of computer uses and limitations.

NATIONAL COUNCIL OF SUPERVISORS OF MATHEMATICS
POSITION PAPER ON
BASIC MATHEMATICAL SKILLS

INTRODUCTION

The currently popular slogan "Back to the Basics" has become a rallying cry of many who perceive a need for certain changes in education. The result is a trend that has gained considerable momentum and has initiated demands for programs and evaluations which emphasize narrowly defined skills.

Mathematics educators find themselves under considerable pressure from boards of education, legislatures, and citizens' groups who are demanding instructional programs which will guarantee acquisition of computational skills. Leaders in mathematics education have expressed a need for clarifying what are the basic skills needed by students who hope to participate successfully in adult society.

The narrow definition of basic skills which equates mathematical competence with computational ability has evolved as a result of several forces:

1. Declining scores on standardized achievement tests and college entrance examinations;
2. Reactions to the results of the National Assessment of Educational Progress;
3. Rising costs of education and increasing demands for accountability;
4. Shifting emphasis in mathematics education from curriculum content to instructional methods and alternatives;
5. Increased awareness of the need to provide remedial and compensatory programs;
6. The widespread publicity given to each of the above by the media.

This widespread publicity, in particular, has generated a call for action from governmental agencies, educational organizations, and community groups. In responding to these calls, the National Institute of Education adopted the area of basic skills as a major priority. This resulted in a Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio, in October, 1975.

The National Council of Supervisors of Mathematics (NCSM), during the 1976 Annual Meeting in Atlanta, Georgia, met in a special session to discuss the Euclid Conference Report. More than 100 members participating in that session expressed the need for a unified position on basic mathematical skills which would enable them to provide more effective leadership within their respective school systems, to give adequate rationale and direction in their tasks of implementing basic mathematics programs, and to appropriately expand the definition of basic skills. Hence, by an overwhelming majority, they mandated the NCSM to establish a task force to formulate a position on basic mathematical skills. This statement is the result of that effort.

RATIONALE FOR THE EXPANDED DEFINITION

There are many reasons why basic skills must include more than computation. The present technological society requires daily use of such skills as estimating, problem solving, interpreting data, organizing data, measuring, predicting, and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills. In recognition of the inadequacy of computation alone, NCSM is going on record as providing both a general list of basic mathematical skills and a clarification of the need for such an expanded definition of basic skills.

Any list of basic skills must include computation. However, the role of computational skills in mathematics must be seen in the light of the contributions they make to one's ability to use mathematics in everyday living. In isolation, computational skills contribute little to one's ability to participate in mainstream society. Combined effectively with the other skill areas, they provide the learner with the basic mathematical ability needed by adults.

DEFINING BASIC SKILLS

The NCSM views basic mathematical skills as falling under ten vital areas. The ten skill areas are interrelated and many overlap with each other and with other disciplines. All are basic to pupils' development of the ability to reason effectively in varied situations.

This expanded list is presented with the conviction that mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics. The ten components of basic mathematical skills are listed below, but the order of their listing should not be interpreted as indicating either a priority of importance or a sequence for teaching and learning.

Furthermore, as society changes our ideas about which skills are basic also change. For example, today our students should learn to measure in both the customary and metric systems, but in the future the significance of the customary system will be mostly historical. There will also be increasing emphasis on when and how to use hand-held calculators and other electronic devices in mathematics.

Appendix 4. Time Distribution Specifications in a Large City School System (From the planning book issued to teachers.)

ELEMENTARY SCHOOL WEEKLY TIME DISTRIBUTION CHICAGO PUBLIC SCHOOLS

EIGHT CURRICULUM AREAS		WEEKLY TIME DISTRIBUTION IN MINUTES BY GRADE							
		1	2	3	4	5	6	7	8
LANGUAGE ARTS* — Reading, Literature, Oral and Written Composition, Grammar, Spelling, Handwriting, Library	MAX.	900	865	700	600	600	550	440	440
	MIN.	850	800	625	500	500	400	315	315
MATHEMATICS —	MAX.	100	125	225	225	225	225	300	300
	MIN.	50	100	150	150	150	150	200	200
SOCIAL STUDIES — History, Geography, Civics	MAX.	150	175	200	225	275	280	300	300
	MIN.	125	125	150	200	200	200	200	200
SCIENCE —	MAX.	100	100	150	175	175	225	300	300
	MIN.	50	50	125	150	150	175	200	200
FINE ARTS — Art, Music	MAX.	150	150	150	150	160	160	165	165
	MIN.	100	100	100	100	80	80	80	80
HEALTH AND PHYSICAL EDUCATION — Gymnasium, Health, Safety, Change of Class, and/or Recess Activities	MAX.	250	250	250	250	250	250	250	250
	MIN.	175	175	175	175	170	170	170	170
PRACTICAL ARTS — Home Economics, or Industrial Arts	MAX.							200	200
	MIN.							80	80
or Home Mechanics	MAX.							120	120
	MIN.							80	80
INTRA-CURRICULUM AREAS — Home Room Guidance, Student Council, Clubs, Assemblies	MAX.							165	185
	MIN.							120	120

*Where English and another language are taught, the maximum time for Language Arts is to be used.

Use of the above time schedule is intended to implement the program contained in the teaching guides and their supplements.

Subdivisions of the subjects listed above and detailed description of the content to be taught are in the respective supplements to the teaching guides.

Suggested time allotments are to be used to plan schedules that will help make it possible to develop the program in accordance with the needs of the children in each class.

First consideration must be given to proficiency in the Language Arts, Social Studies, Mathematics, and Science, with priority given to Language Arts.

In every case, the schedule should equal 1575 minutes per week.

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A Reaction by Jacob Kounin*

I never know what a critique is supposed to be. We might call it a formalized projective test. And, as you probably know, the degree to which there is projection from within the person rather than perception of what is on the outside depends upon the differentiation and complexity of the field; the more differentiated the field, the more selectivity is involved, and the more ambiguous the field the more there is a projection. And since the concept of environment is both differentiated and ambiguous, my comments on Bell's paper are more projective than perceptive.

I am not going to be as generous as Bell about defining an environmental variable as anything outside a person that might affect behavior. Rather, I will follow the Barker and Gump Kansas group in psychological ecology and define environment as a set of behavior settings, as ethologists and ecologists do. They talk about habitats and about environmental niches; they do not talk about personality of occupants. An activity setting has, in their concept, very clear requirements to reduce not only the differentiation but the ambiguity of what the environment is.

I think that somewhere along the line we have to say environment is X or non X. And some of these things that Bell calls environment are not environment, at least not in the behavior setting sense of the word.

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*Jacob Kounin is a professor of education at Wayne State University.

Whether teachers are trained at Michigan State University or Wayne State University, whether they are male or female, or whether they are friendly or unfriendly is irrelevant to what a behavior setting is. Admittedly, personality properties may affect behavior; this is the odd part about it. Raquel Welch could come into this room wearing a low cut gown and she would definitely influence my behavior. But I would not call Raquel Welch, even though she influences my behavior, a part of this behavior setting.

Events occur that are not a part of what one might call a stable behavior setting such as a classroom, library, or arithmetic lesson. It doesn't matter whether they affect behavior or not; to call everything outside the person an environmental variable leads to thousands of paths and cul-de-sacs. Not everything that affects behavior should be identified as part of the environment.

What then are the basic characteristics of a behavior setting such as found in the classroom? To begin with, the classroom has a physical milieu and props -- tables and chairs, books, art supplies, or mathematics materials -- and it is located in a particular place. In addition, a behavior setting has a standing pattern of behavior; there are certain kinds of behaviors that go on in a particular setting. This standing pattern may be thought of as a program, a curriculum, or simply as what goes on in the behavior setting. In the classroom, teachers explain, question, and call for reciters; pupils recite, write, construct collages, and so forth, and these are appropriate behaviors and standing behavior patterns for this setting.

In behavior settings there is also harmony -- or a synomorph, as Barker calls it -- between the standing behavior pattern and the milieu. In one way or another, the arrangement of chairs, pencils, and books fits with the program. Finally, the setting has a temporal and spatial boundary.

I do not think we need interdisciplinary perspectives on behavior settings because one of the characteristics of behavior settings is that they are pre-perceptual. There are certain realities that exist outside a person and behavior settings are among those realities; that is,, they exist independent of any person's perception, and I think a mathematician, a sociologist, a psychologist and an educator would see a setting in the same way. Also, and I think this is integral to an understanding of environment or behavior setting -- the person is surrounded by the behavior setting; topologically speaking, the person is inside the behavior setting. Behavior settings, with their standing behavior patterns, emit certain signals that in some way tell people what to do and when to do it. For example, a symphony concert can be looked at as a signal system; regardless of sociopolitical climate, regardless of community atmosphere, and regardless of ethnic makeup of the population within that setting, behavior that takes place in symphony concerts is pretty much the same. I found it the same in Albert Hall in London, in the Mann Auditorium in Tel-Aviv, in the Ford Auditorium in Detroit, and I suspect it would be the same at Michigan State or San Francisco. The behavior setting of a symphony concert somehow shapes the behavior of its occupants -- performers and listeners alike -- through a series of standard signal systems:

gongs or lights going off, concert master striking an A note, conductor facing audience, conductor facing orchestra and waving baton, etc.

I would propose that when we look at school environment we begin to discriminate and concentrate on the particular signal systems emitted by the behavior settings found, let us say, in the teaching of mathematics. These signal systems, which can be described in terms of degree of continuity, sources of signals, insulation, and protection from external inputs, produce particular kinds of behaviors on the part of both the teachers and the kids. I think we must look at this in a naturalistic way, as do the ethologists or ecologists observing an organism or animal in a particular habitat, if we are to learn anything about school environments and how to run them.

Bell said my research on classroom management is a good model because at one point I had the guts to shift my perspective when I found I was wrong. I would like to correct myself, because I have shifted perspective again since then. I do not talk about managerial styles as such; I do talk about what a teacher has to do in order to make a particular behavior setting work. The teacher must work in accordance with the signal system of the activity that is going on at any particular time. The steps taken to induce students to do seatwork, for example, are quite different from those necessary to implement a successful recitation. In seatwork there is no need for group focusing, maintenance of momentum or pace progress; the kids pace themselves according to signals and effects of what they do. There must be something about the individual seatwork that not only produces changes, but also signals that there is more to do.

Suppose, for the sake of clarification, that a preschool teacher has students make collages by pasting pictures of desserts on paper. The cutting produces a change which signals the kids to paste and then the pasting produces a change which signals them to cut more, and so on; the result is a continuous system. There is a difference, however, between making a collage of desserts and, say, pasting squares, triangles, or circles. In making a dessert collage the system says there is more to do because everybody knows there is more than one kind of dessert; but in the pasting of circles or triangles in a primitive geometry lesson in preschool, there is nothing to signal further work once the pasting is done. So to run individual construction formats (in a continuous and insulated system), the teacher must not only provide the materials but also plan an activity which will say that there is more to do -- that the task is not finished when there is only one thing done.

A different signal system is involved in a teacher demonstration, where the signals come from central emitters, the teacher and his/her props. A good central emitter produces a continuous system without many lags or lacunas. We find that a continuous system produces more task involvement with the same children and the same teacher than a discontinuous system; so what it takes to run a discussion or demonstration is maintenance of continuity in the signal system. If we see that teacher X gives a good demonstration resulting in high involvement with the lesson but later provides a poor demonstration with the same children and the same type of activity structure, we might ask what difference there was between the high and low task involvement lessons. Obviously, the teacher is the same personality, the kids are the same population, and the physical milieu is the same; but we would find that the teacher demonstration

lesson that broke down was the one that slipped into a child discussion format. A child discussion format is one in which the children get signals from other students and the teacher and his/her props, rather than from the props and teacher, alone.

I don't know if I have clarified the signal system, but I think we might try to get a taxonomy of activities for training of teachers. We might attempt to determine the dimensions of signal systems and the related activity requirements -- the props, the physical milieu, the standing behavior and the fit between the props, the physical milieu, and the standing behavior. If we know that, we can show teachers what they need to do to conduct different types of lessons.

If we follow this notion of behavior setting we restrict our notion of environment, as is necessary. It is good to have generality, but it is also important to have a difference between X and non X. If we are going to list properties of occupants, modality of material, teachers expertise, content objectives and pedagogical philosophy as environments, I think we are going to have too much to look at and we are likely not only to get trapped in a cul-de-sac, but to go in an unrelated direction.

A Reaction by Robert Hill*

In reacting to Max Bell's paper and presentation, I wish first of all to remind Bell that while there are self-contained classrooms, there are also many variations on this theme -- variations which may have an impact on learning (e.g., departmentalized schemes, Joplin plans, team teaching).

From my experience as a teacher, I have noticed an unbelieveable number and range of variables which influence teachers and learning. Even something like a custodian failing to clean a room after a messy art project can not be ignored. Instead of bypassing these environmental concerns, as Bell suggests, we should identify them and deal with them in long-term studies. In addition, I support Bell's recommendation that research be conducted to determine the current status of classroom environments.

Concerning planning, Bell mentioned that teachers make few if any plans, relying instead on intuition and experience to manage classroom situations. I believe that intuition and experience can be effective; teachers make instantaneous plans based on student reactions to deal with problems as they occur. Furthermore, I would suggest that many teachers do plan, but in ways not always expressed in detailed, written

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form. Plans made at the end of the day are based on what happened during the day and what is expected to happen the following day. One area for study might be the impact of institutional variables on plans of this kind.

It is my contention that teachers can control what happens in their classrooms. Perhaps this is true only for the "better" teachers, and maybe these teachers exercise control only in the sense that they understand the technology of teaching; however, this is contrary to my belief in intuition and experience.

If learning environments need to be altered -- and I agree that they do -- changes should be implemented in a systematic way to fit planned instruction. I am convinced that there is not just one particular type of environment which will produce good results, but that there are many different environments needed to facilitate a plan. Since classrooms probably will not be substantially altered in the foreseeable future, however, we should view this as a long-term research goal.

In reference to management style, I believe it would be very difficult to identify the best or most desirable forms of management style for all classroom situations. Instead, researchers need to identify the different ways teachers effectively handle the classroom and determine the personality types that can best handle particular styles.

Concerning the three suggestions Bell makes for neutralizing the more general impact of the school environment, I would like to make this additional recommendation: include researchers of different disciplines in a united research effort. Much could be gained from the unique views and perspectives that would be brought to research on teaching by re-

searchers from different disciplines.

Bell's recommendation that research should be conducted within existing classrooms and should involve teacher collaborators will find great support among teachers. Teachers often believe that they perform many trivial tasks in the name of educational research, and they would be glad to know they are finally being considered as important contributors in the research process.

I would also like to point out that Bell's references to the small amount of time students actually spend with mathematics might not be accurate. An informal assessment I was involved in recently, was a study of how 100 students used mathematics, thought about mathematics, and saw adults using mathematics. Surprisingly, it was discovered that these students were all deeply involved with mathematics in one way or an other. Perhaps we should consider homogeneous student groupings based on the interests and needs of the learners.

Along these same lines is the question of students' time-on-task involvement. What's needed here is a better definition of "task." Perhaps there is a hierarchy or division of tasks, with each type of task producing a very specific type of involvement. Educators need to understand this hierarchy.

In conclusion, I would like to support the main categories of research suggested by Bell. Within these categories, however, consideration must be given to grade levels, as this may affect the emphases, methodology, and goals chosen for research. In addition, with decreasing enrollment in schools of education, it is important that this research be focused on improving the practices of inservice teachers.

Discussion*

Bell: Environmental variables seem to be pretty much a matter of definition, though I am certainly willing to reduce their complexity. For instance, Kounin, would you say that the size of the learner group is an irrelevant variable?

Kounin: Barker would say that the size of the group depends upon the number of constituent performances and the ratio of necessary performances to populations. If there are only two performances, then the optimum size of the group is two; if there are 20 performances, the optimum size is 26. The performance to performer ratio is an important determinant of what we sometimes call the independent variables morale. Morale is higher when there are not too many performers for the number of performances available. This is in essence the difference between large and small schools. As an example, a certain play would require the same number of performers in a large school as it would in a small school, but the ratio of performers to performances available is quite different. In a small school, morale is higher and people are more friendly since a larger ratio of them are involved in the production. I would say that the size of the group as an environmental variable depends upon the nature of the activity. For a symphony concert, it doesn't much matter if there are 500 or 2,000 people in the audience, but in a discussion group size is much more important.

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*This discussion, based on audiotapes of the conference, has been paraphrased and condensed to include only the key points.

Bell: You seem to be saying that simplifying the situation in order to get a handle on it is absolutely essential. I agree.

Kounin: Yes, we can simplify it by using "pure" cases, or we can simplify it by developing clear definitions.

Bell: I think the important question about environmental variables is not whether we decide to classify them as environmental variables, but whether they are relevant simply as variables. The real questions are: Is existence and use of time something that has to be attended to? Is the size of the learner group a relevant thing to look at, and so on.

Shroyer: Another thing to decide is whether to look at whole classrooms as behavior settings or to look for behavior settings within the classroom and identify those. There can be as many as two or three behavior settings in one lesson.

Kounin: Yet, what impresses me is the degree to which classrooms are alike.

Shumway: I appreciate that Bell's paper has given us some specific tasks we ought to be concerned with.

Cooney: I would like to react to Max's statement that we should begin our research more or less from scratch. It seems important when starting out in research to try to build on existent research wherever possible. The results of small scale studies can be additive under certain conditions.

I think we also need to consider several additional questions. For

instance, are there pedagogical variables which can be better defined in terms of mathematical content than by other content? What are areas of research that are unique to the teaching of mathematics?

Fitzgerald: I am concerned about our concern for standard situations producing standard kinds of behaviors. I think that if there is one thing characteristic of mathematics teaching at all levels, it is the attempt to produce standard behavior in students. We tend to induce standard kinds of behaviors so that they can form standard ways to approach standardized examinations. Perhaps the direction of our suggestions for improvement should be towards producing non-standard behavior.

Collier: One thing that seems important to me is that most research right now separates what a student does know from what he/she doesn't know. These seem to be two extremes. Somehow we ought to find out what is happening in math education when kids almost know something -- when they never quite reach it. What is happening in those situations? It seems to me that those who are in the middle between understanding and not understanding constitute the majority and should be given more attention.

TEACHER EDUCATION IN ELEMENTARY MATHEMATICS:

A FRAMEWORK FOR NEEDED RESEARCH

John F. LeBlanc

Abstract

LeBlanc states that a key factor in school reform is teacher education but that research has generally neglected this area. Math educators and educational psychologists must work together to generate, interpret, and disseminate educational theories that will prepare teachers effectively. Presented here is a framework for research in teacher education in mathematics which is organized around three basic components: preparation in mathematics (content courses), preparation in mathematical pedagogy (methods courses), and experience in the school or classroom. Each component is discussed and questions posed for potential research.

About the Author

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Teacher Education in Elementary Mathematics: A Framework for Needed Research

John F. LeBlanc

Introduction

Educational reform, which over the years has centered on various educational factors, has now focused on teacher education. Efforts have been newly directed at school reorganization plans, classroom management ventures and, most recently, curriculum reform. A working group of the Institut für Didaktik der Mathematik at Bielefeld established the following thesis as the basis for their research and development work: "Teacher education is a key factor of educational reform . . . . In the last few years the importance of the teacher has increasingly become the focal point in the processes of reforming schools and instructional methods" (Mies, 1975, p. 1).

Teacher education once relied solely on the master and apprentice model. A prospective teacher was prepared by working with, observing, and imitating a practicing teacher. This teaching preparation was broadened through the establishment of normal schools, where subjects such as orthography, reading, writing, English grammar, geography, history, and arithmetic were taught. Still, the predominant feature of teacher preparation was the school experience component. Normal schools were often housed in the elementary school or at least attached to an elementary school called the laboratory school (much the same way as our

medical schools are located in or associated with a hospital). As normal schools grew in number and size, many were converted to teacher training colleges, and some universities established schools of education in which elementary teacher preparation constituted a portion of the program.

Accompanying the move of teacher education into the university was an increase in the number of educators whose function it was to study education from a scientific point of view. The laboratory school was to be not only a training site for prospective teachers, but also a place where educators would have an opportunity to study learning. Such a laboratory would allow educational theoreticians to observe and experiment hand-in-hand with teacher education personnel; this experimentation was intended to result in improved teacher education and effective child learning.

Over the past 30 years, the normal schools have disappeared, and the teacher training colleges created to replace them have been converted into universities. Until recently, a substantial proportion of the universities preparing elementary teachers have had laboratory schools. The reasons for the demise of the laboratory schools in the 1960's and 70's are simple. In the judgment of university officials, the promise of studying the educational process was not being fulfilled by schools of education, and the cost of operating laboratory schools was high. The dream of having an academic community analyzing teaching so that teacher preparation could be more scientific was a dream that ended for many with the closing of the laboratory schools.

While the administrative decision to close laboratory schools

has not been a popular one among many teacher educators, university officials as well as others have a lack of faith in academically-oriented schools of education. The university community generally regards the school of education as a less-than-equal academic entity. Our own education students often criticize our courses as having less-than-sound content. Practicing teachers pointedly label their own teacher education insufficient and irrelevant. Teacher organizations are demanding that practicing teachers play a more important role in the education of teachers. More recently, parents and the general public have rebelled at teaching practices which, in their opinion, have neglected the basics. Accompanying this back-to-basics move has been a clamor for teacher education reform. This clamor and the resulting activity has not been instigated by teacher educators; Performance Based Teacher Education (or Competency Based Teacher Education) has been primarily pushed by state boards of education or state legislators (NACOME, 1975, p. 98).

While the function of teacher education in the university is -- in its simplest terms -- to prepare teachers to teach effectively, there is an inherent additional commitment for the school of education. This commitment is to generate, interpret, and transmit theories related to effective teaching. Without the generation and dissemination of educational theories supported by appropriate research, one might well question the setting of teacher education in a university rather than in a school under the master and apprentice model. Perhaps the various divisions of the school of education such as teacher education and ecucational psychology are counterproductive and self-serving if there

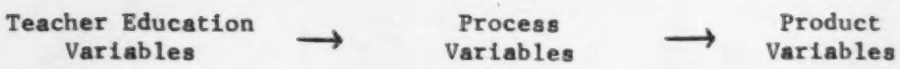
is not active communication and cooperation between them. Mechanisms must be found so that psychologists and teacher educators work together to generate sound educational theory and, in turn, a sound teacher education program.

The assumption that the function of a teacher education program is to prepare teachers to teach effectively presupposes that teacher educators know the factors and processes which contribute to effective training. Such a supposition is, of course, both simplistic and untrue. While it may not be quite the case of the blind leading the blind, there is reason to be apprehensive about our objective integrity in training teachers. In our most candid moments we must admit that our own practices in teacher education rely heavily on an experiential basis (our own or that of others) and plausible constructs rather than on research evidence.

A Framework for Research in Teacher Education in Mathematics

Efforts to gather research evidence must be intensified so that teacher education can rest on more objectively-based tenets. The proof of any educational theory must be, in the final analysis, judged by the effectiveness of child learning. Teacher educators and educational psychologists must work together in some "laboratory" school setting to produce a set of research-based tenets for teacher education. A framework for research which ties teacher education to pupil learning is provided by Dunkin and Biddle (1973, p. 38). A simplified form of this framework (Figure 1) should be kept in mind as this discussion continues.

Figure 1



Teacher education in mathematics has three generally accepted components: (1) preparation in mathematics (content courses); (2) preparation in mathematical pedagogy (methods courses); and (3) school experience. In this paper, the framework for research in teacher education in mathematics is organized in the following manner: Each component of teacher education in mathematics is briefly discussed, and some specific aspects of each component are identified. Following the discussion, some questions are raised which suggest needed research.

Preparation in Mathematics

Twenty years ago, essentially no preparation in mathematics was required for prospective elementary teachers. The efforts of the Committee on Undergraduate Programs in Mathematics (CUPM) of the Mathematical Association of America, the National Council of Teachers of Mathematics (NCTM), and individual mathematics educators have resulted in increased preparation in mathematics for elementary school teachers. The CUPM recommendations published in 1963 (and revised in 1966 and 1971) suggest that a minimum of nine semester hours of content be required for the preparation of elementary school teachers (CUPM, 1971). This recommendation, accompanied by rather specific syllabi for the courses, is based on opinions gathered at several conferences aimed at identifying the needed preparation in mathematics for elementary school teachers. This emphasis on the mathematical preparation of future teachers rests on the assumption

that an increased knowledge base will lead to more effective teaching. A number of questions can be posed as a result of this assumption.

Teacher education. As a result of greater course work in mathematics do teachers:

- possess greater computational skills?
- exhibit greater understanding of mathematical concepts?
- have more confidence in their ability to teach mathematics?
- have a more positive attitude toward mathematics?

Teacher education ⟶ process. (1) Do teachers with increased course work in mathematics teach with greater clarity? (2) Are they:

- better able to sequence arithmetic skill development?
- better able to develop initial concepts in geometry, probability and statistics or in other topics which CUPM has recommended as needed preparation?

Teacher education ⟶ product. (1) Do children learn more effectively from teachers who have increased course work in mathematics? (2) In particular, do the children taught by teachers with increased course work in mathematics:

- show increased learning in basic skills?
- demonstrate increased understanding of mathematical concepts?
- exhibit increased performance in arithmetic applications?
- show an increase in comprehension of topics such as geometry or probability and statistics?
- exhibit a more facorable attitude toward learning of mathematics?
- show more confidence when working with mathematics?

Are there any relationships between teacher mathematical preparation levels and the ages of the children taught?

On the surface, it seems foolish to question the assumption that increased teacher preparation in mathematics would result in favorable process and product variable changes. Yet, Fey (1969) reported that there was, "Almost no evidence to support or deny claims of correlation between teacher knowledge of mathematics and classroom effectiveness" (p. 80). The assumption that greater mathematical preparation is desirable generates another set of questions for teacher education:

- What should be the content of the mathematics courses?
- How should these courses be taught?
- Should the content courses be separated from the methods courses or should the two components be integrated?

In considering the last question it should be noted that while there seems to be strong support for such an integrated approach, there has been little research evidence behind it (NACOME, 1975, p. 87).

Dodd (1974), in her doctoral study, compared the performance of children taught by three groups of preservice teachers. The first group was trained with the Indiana University Mathematics Methods Program (MMP), the second trained in standard, separate content and methods courses, and the third trained in a professional year program (integrated across several subject area methods courses). Significant differences in pupil performances over an equivalent fractions test were found to be in favor of the pupils taught by the MMP-trained preservice teachers. Further, the preservice teachers trained in MMP were observed as possessing a higher degree of competencies associated with clarity, questioning, and involvement, all of which were highly correlated with pupil scores in this study (Dodd, 1974).

Preparation in Mathematical Pedagogy

Discussion in this section will be organized around six categories of objectives which I suggest comprise the content of a methods course in elementary mathematics. These six categories are:

1. Point of view in teaching mathematics.
2. Knowledge related to mathematics and curriculum.
3. Instructional strategies.
4. Instructional materials.
5. Evaluation.
6. Professional aspects.

Each category will be discussed, then followed by some appropriate questions related to teacher education and the process and product variables.

1. Point of View in Teaching Mathematics

Mathematics educators frequently impress upon inservice and preservice teachers the need to approach mathematics from a "real-world-to-mathematical-symbols" point of view. According to this approach, a teacher should introduce each new topic in mathematics using concrete or pictorial aids -- some type of real world referent. Some mathematics educators encourage preservice teachers to teach mathematics in a highly developed, hierarchically-sequenced point of view (as might Gagne), while others encourage their preservice teachers to teach using a problem-solving point of view (as might Bruner).

Mathematics educators often try to have teacher candidates put the role of arithmetic skills in perspective with other mathematical content. An overemphasis on arithmetic skills is presented as being a short-sighted view of mathematics and, indeed, harmful to the children's mathematical development.

Teacher education ⟶ process. Some basic questions arise related to the point of view or philosophy in teaching mathematics: (1) Do teachers trained by an educator with a particular point of view reflect that point of view in classroom instruction? (2) Do teachers who have been instructed to overemphasize arithmetic skills demonstrate a different emphasis in their instruction?

Teacher education ⟶ product. (1) Do the students of teachers trained with a specific point of view learn more effectively than those taught by teachers trained with no specific point of view? (2) Do the students of teachers trained not to overemphasize arithmetic skills show equal achievement in these skills? Do they show greater gain in other mathematics learning?

The answers to the above questions and the refinements of those questions depend heavily on some much-needed process/product research. For example, it appears that some topics might be learned more effectively if presented in a hierarchically-sequenced order, while others might better be presented in a problem-solving mode. The approach taught to preservice teachers might well depend on the topic to be learned or even on the age of the learner.

Many mathematics educators feel that the classroom teacher's point of view or philosophy toward mathematics instruction (or even toward the role of mathematics itself) is an important factor in learning. They argue that the point of view a teacher holds could affect his/her entire classroom instruction and the resultant child learning. Thus, mathematics educators believe it is necessary to know just what impact a point of view has on learning and -- assuming this impact is substantial -- which point of view is most favorable. Finally, we must be concerned about how to

train teachers in such a way that they acquire the appropriate point of view.

2. Knowledge Related to Mathematics and Curriculum

The mathematical content which teacher candidates have studied prior to enrollment in a methods course often includes the foundations of the content taught in the elementary school. The relationship between college mathematics and school mathematics often is not apparent to the preservice teacher, yet mathematics educators view this relationship as an important part of teacher preparation. To further complicate matters, mathematics educators often find that when they try to relate college mathematics and school mathematics, they must re-teach the mathematics content itself since many students seem to have "forgotten" or claim they never did "understand" the content of the courses they had passed.

It seems reasonable to demand that teachers be able to do the mathematics or perform the computational skills which they teach to children. Some preservice teachers actually have difficulty performing certain computational skills which are found in the elementary curriculum, and mathematics educators are faced with the decision of helping them learn or relearn basic computational skills.

A knowledge of the typical sequence and scope of an elementary school curriculum is also necessary for a teacher. The sequence is important in terms of the decisions a teacher must make in organizing instruction; the scope is important in making decisions about the relative emphasis a topic should receive or how worthy that topic is of further study.

Another significant area of knowledge which mathematics educators include (or wish they could include) in teacher education programs is research findings on children's cognitive development in mathematics.

Some mathematics educators have made concerted attempts to interpret the research findings of Piagetian studies or studies related to Dienes and other theorists to suggest implications for classroom instruction.

Teacher education. (1) How can mathematical content and/or pedagogical courses be fashioned so that the relationship between school mathematical learning and college mathematical foundations is apparent to the preservice teacher? (2) Can preservice teachers learn or relearn basic computational skills in a methods class? (Or should such a set of skills be a prerequisite?) (3) Referring back to a previous discussion, should the mathematical courses and the pedagogical courses be taught separately?

Teacher education → process. (1) Do teachers who can relate school mathematical learnings to their college foundational mathematical preparation prepare instructional lessons which reflect greater clarity? (2) What effect does educating teachers on research in children's cognitive development in mathematics have on teachers' instruction? (3) Are teachers who have knowledge of the sequence of a mathematical topic better able to organize instructional activities for children? (4) Are teachers who know the scope of the curriculum able to reflect appropriate emphases on important topics in their instruction?

Teacher education → product. (1) Do children taught by teachers with a greater understanding of the relationship between college mathematics and school mathematics learn more effectively? (2) Is there a difference in the performances of children taught by teachers who are able to carefully sequence a topic? (3) Do children learn more effectively when taught by teachers who have been trained in the research results related to children's cognitive development in mathematics?

In connection with the last question, Turner (1963) found a high correlation between teacher performances on a test which included a sequencing task and child achievement. This sequencing task involved the ordering of seven division examples from simplest to most difficult. Knowledge of the curriculum sequence, the teacher's ability to sequence a topic, and the relationship between a teacher's ability to sequence and child achievement is a particularly fruitful area for research. Much of the learning of the basic skills in arithmetic seems to be dependent on previous learning. Thus, much of the teacher's task seems to be related to identifying prerequisite skills and carefully teaching and building on each. The questions related to teacher preparation for this particular task clearly deserve priority attention.

3. Instructional Strategies

Mathematics educators place a heavy emphasis on preparing teachers to use different models for introducing mathematical topics. For the arithmetic operations the models include the set model, the number line model, the region model, the array model, and the "function machine" model. A knowledgeable acquaintance with these models is viewed as important for preservice teachers. Along with this acquaintance, a prospective teacher is trained to be flexible in the selection of the models so that he/she might be able to possess and utilize a repertoire of these models in his/her classroom instruction.

Helping prepare teachers to carry out instructional tasks also includes training in the preparation of lessons for individual differences. These lessons may be introductory, maintenance, review, re-teaching, enrichment, or remedial. To help teachers learn to prepare lessons, research findings regarding such issues as the use of examples and non examples,

the role of questioning, the emphasis on pupil interaction, and so forth must be brought to light. Research reviewed by Rosenshine and Furst (1973) indicates that some variables (nine are listed) of instructional strategies do correlate with greater learning effectiveness.

Teacher education. (1) Can simulation experiences (using videotape or other means) be used to better prepare teachers to recognize and construct appropriate instructional strategies which take into consideration the many variables involved in an actual classroom setting?

Teacher education → process. (1) Can teachers be prepared to include in their instruction some of the known variables correlated to effective learning? (2) Are teachers able to vary their instructional strategies depending on the ability or age of children, type of lesson, and nature of the content? (3) Do teachers prepare different lessons depending on children's abilities and the nature of the learning (introductory, maintenance, etc.)? (4) Does the manner of preparation (e.g., simulation, lecture-discussion) make a difference in the richness of instructional strategies which teachers employ? (5) Do teachers actually use the instructional models and strategies taught in methods class in their classroom instruction?

Teacher education → product. (1) Do teachers having a greater repertoire of strategies for the models (sets, number line, etc.) produce more effective learning in children?

As with other categories of preparation in mathematical pedagogy, the mathematics educator is dependent on answers to questions which are the domain of process to product variables. What models produce the most effective learning for children? Does the age of the child or the mathematical topic make a difference? Does the use of several models

aid or produce confusion in learning? In particular, mathematics educators are interested in research related to the role of models associated with mathematics instruction. In many methods courses the total course is organized around the use of models in developing mathematical concepts in children.

4. Instructional Materials

Materials constitute an important part of the teaching task. In the teacher's preparation, attention is given to the kinds and usage of a variety of materials such as textbooks, visual aids, and classroom aids (rulers, abaci, charts, etc.). In particular, the greatest attention is given to the classroom aids. These aids are studied in the context of developing skills and concepts with children. Often, these aids serve to represent a model for the concept being introduced (e.g., an abacus for place value). Many mathematics educators spend a considerable amount of time developing games, puzzles, and other activities on the assumption that concrete aids manifest a point of view in learning mathematics (i.e., concrete to abstract) and also induce in children a motivation to learn.

Teacher education ⟶ process. (1) Do teachers actually use the materials introduced to them in the methods course? (2) Does it make a difference whether the introduction of materials includes actual preservice teacher involvement (rather than passive observation) with the materials? (3) Do teachers use materials solely for motivation or fun, and/or do they use them in connection with the development of a concept or skill?

Teacher education ⟶ product. (1) Do the students of teachers trained in methods courses which emphasize material aids exhibit:

- greater achievement?

- better attitude?

As with other categories in the methods course, many of the decisions for emphasis should rest on the process-product research. For example: Does the achievement of children improve with the use of materials? Does the type of concept or age of the children make a difference in achievement when aids are used?

5. Evaluation

The emphasis on diagnosis of mathematical difficulties has increased in recent years. While evaluation includes topics such as the use and role of standardized tests, teacher-constructed quizzes, homework, and non pencil-paper types of evaluation, the major emphasis in evaluation today is on pupil-error diagnosis. Concomitant to the increased emphasis in the diagnostic preparation of teachers has been the formation of several diagnostic clinics in mathematics across the country. A primary goal of these clinics is to give teachers expertise in diagnostic and remediation procedures. The emphasis in such clinics is on "preventive" teaching; that is, teaching to prevent errors which children might make. A fast-growing organization devoted to the problems of diagnosing and remediating mathematical errors has drawn considerable interest among mathematics educators. The annual meeting of this organization during 1977 drew 500 participants, most of them concerned with some phase of teacher education in mathematics.

Another category of evaluation is the mathematics achievement of children with special needs. Mathematics educators are having substantial difficulty in training teachers of children with these special needs (special education children, children with learning disabilities, etc.). There has been very little attention given to this population of

teachers by either curriculum developers or researchers in mathematics education. This category of evaluation is one which many mathematics educators feel to be quite inadequate. In particular, there is a strong need for research in this area to provide guidelines for teacher preparation. Several questions follow.

Teacher education ⟶ process. (1) Can teachers be trained to employ diagnostic procedures in their classrooms? (2) Is clinical training an essential or at least a significant variable in the training of teachers to diagnose errors in mathematics? (3) What tasks does a successful teacher of children with special needs employ in his/her teaching? (What tasks does the unsuccessful teacher employ?)

Teacher education ⟶ product. (1) Do teachers with "preventive" teaching techniques produce greater child achievement? (2) Can teachers trained in diagnostic and remediation procedures produce greater achievement in children?

The need for research in teaching mathematics to children with special needs is critical. With very few exceptions, mathematics educators are "flying blind" in the preparation of teachers for these tasks. The preponderance of research in error diagnosis has focused on children in normal classrooms making recognizably logical errors, but the errors made by children in special education do not fall into the standard error categories. Only a massive effort and attack on this problem can improve this situation.

6. Professional Aspects.

Teacher preparation in a methods class encourages candidates to join professional organizations, read journals, and, in general, pursue their education related to mathematics.

Teacher education → process → product. (1) Do teachers who read journals and attend conferences related to mathematics instruction include new ideas in their classroom instruction? (2) If teachers continue their professional growth, does it make a difference in the achievement of the children?

Although three components have been listed for the education of elementary teachers in mathematics, only one -- preparation in mathematical pedagogy -- is common for all elementary teachers. Even this component varies in terms of the objectives and content. Some states do not require any mathematics content courses, and none require a special experience in elementary mathematics. Thus, the importance of research related to preparation in mathematical pedagogy is important because findings in this area might have the greatest impact upon classroom application of research.

School Experience

School experiences for preservice teachers in elementary school mathematics are wide and varied. Educators in general are just beginning to realize and accept the importance of this component in teacher education. When teacher education moved from the normal school to the university and the laboratory schools were closed, less and less emphasis was placed upon school experience. Until recently, it was not unlikely for a teacher candidate's first contact with children to take place in his/her senior year during student teaching. Even today there are very few institutions which require a special school experience focused on mathematics instruction.

On the other hand, it is not uncommon to hear mathematics educators complain that the "good things" which they teach their preservice

teachers go out the window when the student does his/her practice teaching. The influence of the supervising teacher on the future practices of a teacher candidate, according to the educator, is greater than that of the methods instructor. Whether or not this claim is entirely true, it suggests that the master and apprentice model is not completely dead. If such a claim is true, a mathematics educator who teachers preservice teachers in a classroom setting might effect a greater influence on their future practices.

Again, the move of teacher education toward a field base is being pushed by forces other than, or at least in addition to, schools of education. More and more states are requiring field experience as an on-going process in the undergraduate teacher education program. The National Council for Accreditation of Teacher Education (NCATE), in its 1970 Standards for the Accreditation of Teacher Education, requires that laboratory and clinical experience accompany professional courses in teaching and learning theory. This requirement is stated in a separate standard from the one requiring student teaching experience. With this emphasis on school experience the mathematics educator will have a greater opportunity to demonstrate how a point of view should be included in the instruction of children, how a topic might be sequenced, how an aid might be effectively used, etc. The mathematics educator can select those aspects of instruction for demonstration and supervision which he/she believes to be important.

Teacher education ⟶ process. (1) Which mathematics experiences in practice teaching carry over into the teacher's classroom instruction? (2) Does a mathematics educator's supervision of a school experience have a positive impact on the teacher's instructional strategies?

Teacher education ⟶ product. (1) Does school experience during the preservice training of teachers (specifically in mathematics) make a difference in the achievement of children? If so, which experience?

There is little question that school experience will play a greater role in the education of teachers; the question is how should such experiences be planned. If schools of education do not plan these experiences, other groups will. The implementation plans for states enacting Competency-Based Teacher Education laws include the active participation of teachers and teacher organizations. There is some urgency to do careful thinking, planning, and, most of all, research on what school experiences effect the most favorable process and product outcomes.

Some Concluding Remarks

The framework for research in elementary mathematics teacher education just presented is exactly what it's labeled -- a framework. The discussion accompanying the framework does not, except in a few isolated instances, suggest what research priorities should exist. Different populations would undoubtedly suggest a different ordering of priorities. It is generally conceded, for example, that the public wants improved teacher preparation, teaching, and learning in the basic arithmetic skills; research related to that goal would be a high priority for that population. Few mathematics educators, however, would select such a goal as first priority. In fact, mathematics educators often spend considerable energy trying to influence preservice teachers to spend less time on computational skills and more time on other mathematical topics. Thus, research priorities would be different for mathematics educators than for the general public.

Mapping the priorities for research in teacher education d[illegible]s on the priorities of goals in elementary mathematics. Without attempting to resolve the apparent conflict between the public's priorities and mathematics educators' priorities, some considerations ought to be kept in mind when establishing research priorities.

Teacher education should help prepare teachers to perform their teaching tasks effectively. The emphasis in teaching elementary mathematics as reflected by the content in elementary textbooks is predominantly on basic computational skills. Both preservice and inservice teachers recognize and accept this task, and there is little reason to believe that a large percentage of time and effort will not continue to be spent on this effort. It seems reasonable to research questions related to this task if for no other reason than to achieve effective teaching/learning in this area so time can be spent on other topics.

There are several fruitful areas for research in more effective teacher preparation related to basic computation skills instruction. Among these areas of research are questions related to training teachers in sequencing instruction, diagnosing accurately and prescribing appropriately, and selecting and using appropriate models and aids.

Problem solving is one goal of elementary school mathematics high on the priority list for both the public and mathematics educators. How might teachers be trained to teach problem solving more effectively? A first step might be to observe several teachers teaching problem solving. Observers would record the moves of teachers, identifying those moves which seem to produce effective learning and those which

do not. This list of "good" teaching moves could serve as the basis for treatment in the training of teachers. After the training period, an assessment of the instruction by the teachers trained and the resulting child performances in problem solving could provide additional insight into appropriate training techniques. Such a sequence of observation for good moves, training for these moves, and evaluating the training might well serve as a model for research related to many teacher education questions.

Whatever priorities listed, whatever design models for research evolved, research related to teacher education is a critical need in mathematics education. For many mathematics educators the preparation of teachers is a major function. Yet, whether this preparation is for preservice or inservice teachers, very little of it is supported by research. Of the thousands of educational studies recorded and summarized, only a handful relate to teacher education.

Teacher education is a key factor, if not the most important factor in school reform. Schools of education were established to improve the quality of teacher preparation. Divisions of labor were established within the education schools to encourage greater progress in theoretical and practical work. The laboratory school was designed to serve the theoretician and researcher in verifying theoretical constructs; at the same time, the laboratory school served the teacher educator in verifying the effects of training. Over the years the laboratory school has disappeared. In addition, the researcher and teacher trainers no longer work together toward common goals. The Michigan State University Institute for Research on Teaching holds a promise for establishing a

model which would unite teacher educators and psychologists into a cooperative team. This cooperative team would monitor children's learning as a result of teacher education and classroom instruction and would resemble in function the original mission of the university school of education.

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A Reaction by Judith E. Lanier *

The major issues LeBlanc addresses are related to the small amount of valid knowledge we have concerning the content, methods, and school experience necessary for the preparation of teachers. I am sympathetic with LeBlanc's recommendation that we do not just begin (or continue) research in some ad hoc way, but that we identify theoretical frameworks for pursuing such inquiry. When research plans emanate from a conceptual model, however, the strength of the model employed should be rigorously examined so that its limiting factors are recognized. Thus, while I agree that a framework is needed, I want to indicate limitations I see in the one recommended by LeBlanc.

In particular, the linearity of the model he presents appears at once simple and confused. Is it reasonable to expect, for example, that given a certain amount of math content, a course or two in methods, and some particular form of practice, a teacher will be prepared to select appropriate content and instruction for diverse students in diverse contexts and, in turn, show significant learning gains for the students ? This seems to be the assumption of the framework suggested, but I am concerned that many intervening variables are overlooked and their critical interactions ignored.

Let us also examine the suggested model and decide which portion represents research on teacher education and which portion describes research on K-12 teaching. To study teacher education is to study adults being taught to improve their teaching ability--in this case, their math

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teaching ability. Research on K-12 teaching should focus on the study of children being taught to improve their mathematical performance. The K-12 teaching research clearly has implications for research on teacher education, but perhaps we should consider research on teaching children and research on teaching teachers separately, so we might see the distinctions more clearly.

Research in teacher education should give emphasis to the clarity of its dependent variables, i.e., increased emphasis on the expected outcomes of teacher education, rather than more attention to the independent variables of content, methods, and practice. If we would begin our research planning by attending to the outcomes question--how teachers who have differential amounts and types of content, methods, and practice are expected to systematically differ in their teaching -- we might see more direct linkages between the independent and dependent variables. This may not be possible, however, since thinking about outcomes and how to productively describe them is extremely difficult. To my knowledge, the academic world has never succeeded in thinking very long or hard about the goals and purposes of its teaching and how the desired ends could be productively measured. While it is more comfortable to stay at the "instructional treatment" level, I think that research on teaching teachers needs much greater specificity about what teacher education is meant to effect.

Mathematics teacher educators frequently ignore the apparent conflict between the public's desired outcomes (e.g., a society with arithmetic computation abilities) and what they generally prefer to have teachers teach (e.g., creative problem solving). This leaves teachers expecting that they should be teaching one type of content, but receiving pressure

from another group to teach something different. Teachers are generally not prepared for this dilemma, and perhaps we should build such preparation into our programs. My point is that one outcome of a teacher education program might be to produce teachers who can educate the public to the difference between arithmetic and mathematics. With this outcome in mind, there would probably have to be additional requirements in the program beyond content, methods, and practice. Perhaps, for example, political science and public speaking would be necessary. Here, the education of the classroom teacher is extended to that of a change agent for a particular view of mathematics--a view that may not be held by a given community. This outcome might be more appropriately measured by examining teachers changing public attitudes, curricula, and student performance measures to reflect creative problem solving, rather than teachers changing student gain scores on traditional tests.

Another limitation of the framework suggested by LeBlanc is that it focuses heavily on initial preparation or preservice teacher education. While preservice issues should not be forgotten, neither can we ignore the growing need for inservice education. There is clearly a difference in the needs of a person preparing to teach and one who has taught a number of years. Inservice education might range from a one-day workshop or conference to the continuing in-depth work of curriculum development. The framework he suggests does not seem as conducive to inservice teacher education as it does to preservice.

We might also consider ways that teacher education can facilitate research on K-12 teaching. Those of us in the field of higher education frequently refer to teachers as if they were someone other than ourselves. We too teach, and we too are influenced by many of the same things that affect elementary and secondary school teachers. Factors such as limited materials and resources, the amount of available preparation time, class

sizes, and the ability of our learners influence what we do while teaching. Studying our own teaching in teacher education can help us be more sensitive to factors other than teacher knowledge and skill that affect the results of research on teaching. We can compare likenesses and differences in "appropriate" mathematics teaching for not only children, but also adults learning to be professional educators of children. The idea that our own classes can be laboratories for research on teaching is one that has not been explored adequately and should be investigated further.

Finally, an important dissemination relationship must be established between research on K-12 mathematics teaching and that of teacher education in mathematics. Teacher education programs provide an excellent opportunity for validation and dissemination of information about problems and promising findings from K-12 studies of math teaching. It is not just through written documents, but through adaptation of K-12 research results by teacher education programs that we can more effectively share as well as test the external validity of the research knowledge gained.

In summary, let me emphasize my support for LeBlanc's suggestion of using conceptual frameworks to guide our empirical work. But let me advise caution in selection of such a framework. Most of our earlier paradigms have been too simple in assuming that teacher behaviors can be readily traced to pupil outcomes. To hope that teacher education can be linked in a relatively simple or direct way to pupil outcomes will, I predict, bring disappointment. The many interacting and intervening variables that come with diverse learners in diverse instructional contexts seem to require a more comprehensive and complex set of theoretical constructs. Further, the relationship between research on teaching mathematics subject matter to children and research on teaching mathematics teaching to adults should be clarified.

A Reaction by Sharon Feiman*

LeBlanc makes several cogent points about needed research in mathematics teacher education for elementary schools in his paper. As he states, the question of what we should do to prepare teachers to teach mathematics in elementary schools is indeed an open one. Before we try to test the efficacy of particular teacher education practices, we need to first consider these questions: What might an elementary school teacher need in order to teach mathematics? What are the logical requirements of the job? These questions cannot be settled by empirical research. They require conceptual analysis, a task which should be undertaken by more than just mathematics educators. There is reason to be apprehensive not only about the objective integrity of our programs, as LeBlanc points out, but also about their conceptual clarity. LeBlanc recommends that the psychologist join the teacher educator in generating sound educational theories on which to base teacher education practices. Others should participate as well.

If we consider what is involved in seeing that pupils learn math, we might note the following requirements: (1) knowledge of the subject matter and the ability to tailor it to young learners, (2) understanding of the young student, and (3) a conscious seriousness about education as implied by a commitment to "see that pupils learn math." What a teacher

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needs in order to help pupils learn mathematics may cover more territory than what is offered in conventional teacher education programs. LeBlanc mentions, for example, the need for teachers to understand how children develop and how math operations and concepts fit into childrens' views of reality. If in our research we ask questions like, "What is involved in seeing that pupils learn math?", the need for such an understanding becomes clear. If, however, we ask, "What should a teacher-trainer be doing in a methods course?", we may overlook such important aspects of teacher training.

LeBlanc proposes that the criterion for evaluating teacher education practices is what contribution they make to effective teaching. Effective teaching, in turn, must be judged by its contribution to pupil learning. While our ultimate goal is to make the correlation greater between teacher education practices and learner outcomes, there are many intermediate concerns which teacher educators must address. Being a teacher is not coextensive with teaching. There may be some things which it may be important to do in teacher education programs yet which would never pay off visibly in the classroom. Teacher education cannot focus solely on shaping the teacher's behavior in the classroom. The process/product model glosses over individual differences among teachers and students and leaves out contextual factors.

In his discussion of course work in mathematics, LeBlanc poses several questions about the possible relationship between the academic mastery and pedagogical dexterity of the teacher. It is difficult to assess the appropriateness of such questions. How might more course work in math at the teacher-training level affect teachers' understanding, attitudes, and skills in teaching? Fey reports no strong

correlation between teacher knowledge of mathematics and classroom effectiveness. Getting children in classrooms to learn math is quite different from getting teachers to understand math themselves. Knowing one's subject matter seems to be necessary but not sufficient to make one a good teacher.

It seems to be important that teachers have a commitment to teaching math based on an understanding of the forms of mathematical thought and their uses. It is not clear whether this is what LeBlanc means by his phrase "point of view," but his examples suggest that he means the best or favored way of teaching math. Can there be any one approach which will work for all types of math, all teachers and all pupils? I dislike the question: How does one train teachers to acquire the appropriate point of view? A better formulation might be: Which kinds of students seem to learn which mathematical topics by means of which instructional strategies?

I am intrigued by the general notion of a "point of view." The preservice elementary teacher education program at the University of Chicago views teaching as on-going inquiry. The program concentrates on helping teachers clarify what they are trying to teach and why, how their aims fit with the materials selected, and the activities planned. It encourages them to use their classroom observations of what happens as a basis for deciding what strategies to use. The program does not advocate a particular methodology; it tries to foster an experimental orientation.

Some of the most interesting research questions LeBlanc raises about instructional strategies and materials are: Do teachers actually use the models, methods, and materials they learned in their courses? What problems do they encounter? What happens when you teach teachers to

use manipulatives, but the school doesn't provide the materials and there is no accessible teacher center where they can make them?

Several years ago, I conducted an observation study to find out how teachers spent their time in one teacher center. I found that the most frequent activity involved making manipulatible materials and games, and the most frequent kind of talk centered on the construction of such materials, not their use. The center's staff had expected teachers to be more concerned with questions of substance and application. The descriptive data resulting from the study helped staff members reassess their role.

LeBlanc suggests that math educators study highly skillful teachers. David Hawkins makes a similar suggestion, believing that "the best practice exceeds the best theory in quite essential ways." He recommends that we find the very best teachers and engage in close observation and dialogue. Experienced teachers could be valuable colleagues in formulating and investigating problems about the teaching and learning of math in elementary schools. I expect that this kind of collaborative research would also serve as a powerful form of inservice education.

Teachers generally cite practice teaching as the most valuable part of their teacher preparation. Research shows that the cooperating teacher has a greater influence on the student teacher than does the math educator. Perhaps more thought should be given to the organization and monitoring of field experiences. Many problems in teacher education stem from the way programs are organized. The conventional format which separates content and methods and university-based course

work from school experiences may work against the goals of math educators. Before we can figure out "how" field experiences should be planned, we need to determine what they are intended for.

Discussion*

LeBlanc: The act of teaching is human interaction. It is not oriented to a particular content area, though of course, the school does have the function of producing certain academic achievements. One of the things I think we have to find out is what content achievements a teacher is supposed to produce in his/her students -- not necessarily excluding all of the other things that we hope teachers will do.

Lappan: This discussion reminds me very much of a cartoon I saw in Learning magazine about the "bionic teacher." I have the distinct impression that we are trying to build the bionic teacher this afternoon.

Feiman: Our task is not unrelated to building a bionic teacher. And for that reason, we need to develop more precision and clarity about the differences between training and education. What things that we teach prospective teachers can be called training and what things can be called education in a larger sense?

Davis: That's almost an introduction to what I'm going to say. LeBlanc said a moment ago that you will see the act of teaching is human interaction -- the content is secondary. Things like that are what we might want to study, as Dorothy Lee did. Lee studied the education practices of the Dakota Indians and the literacy education of three-, four-, and five-year old Jewish ghetto boys in Eastern Europe prior to World War II. The curricula in both cases were terrible. The Jewish boys were taught

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*This discussion, based on audiotapes of the conference, has been paraphrased and condensed to include only key points.

to read theology in Hebrew (a language they had not heard at home) -- hardly an appropriate subject for three-year-olds. It would be equivalent to having textbooks of quantum mechanics in a Latin class. The students sat on wooden benches. There was no heat in the school building in winter and no ventilation in summer. Everything was terrible. But the kids all learned to read; they loved to read; they often spent the rest of their lives as scholars and looked on reading very highly.

In the case of the Dakota Indians, Lee discovered that all Dakota boys learn to hunt buffalo at about six years old. A six-year-old Dakota Indian must be fairly little and buffalo-hunting has got to be a fairly difficult task. Yet it was achieved successfully without teacher training, without a curriculum, without audio-visual aids. Lee found in every case she looked at what she called "cultural push." The societies in which such things work are determined that they are going to work. Everybody is determined that the methods are going to work. There is no doubt. Nobody wondered, "Is this kid a non-reader?" "Is this kid a non-hunter?" That kid was going to hunt like all kids had always hunted.

It is indeed human interaction going on in the first grade classroom. If the student gets the impression that that teacher believes the content he/she is teaching is important, the student is obviously going to learn it unless, of course, the teacher is worried that the student might not learn it.

Masterson: These are the kinds of things we need to test. The kinds of laboratory programs LeBlanc describes in his paper would give us an opportunity to use teachers to test the efficacy of the models of teacher training that we are coming up with. We have seen that where we

think a particular method will teach the use of Cuisenaire rods or the abacus, it instead produces responses from the students requiring the student teachers to discuss the content in very general terms. To aid in this, we have added psychological components to our courses, having the students read in the area of developmental psychology. We try to phrase methods questions in terms of developmental psychology as well as techniques. When student teachers get into the classroom, things all of a sudden become much more varied; they have to deal with much more difficult responses. For instance, some of the kids may begin throwing the Cuisenaire rods around -- obviously the teacher must constrain the situation more.

We have been forced to re-construct our teacher training course from the point of view of the experiences that are going on in the classroom every day. Instead of questions about Cuisenaire rods, the teacher gets "Can't I work in my workbook now because Mary is two workbooks ahead of me?" Perhaps we need to discuss the use of manipulative devices in our teacher training.

Lesh: I think that we ought to look at good teachers and see what they do. I also like the idea that we ought to consider ourselves to be among those teachers and so look at what teacher trainers do. A good teacher trainer would acquaint teachers with teaching strategies. Yet, those strategies don't seem to cover the variables present in a clinical teaching experience. We're trying to come up with more teaching variables to get a map of our research.

I find myself relying on two different points of view as to how knowledge occurs. One view is that ideas build up one on top of the other, getting bigger and bigger like a brick wall. The other point of

view is that a researcher simply goes into a situation and sorts out what variables he/she can find. Many of the variables that I'm hearing the last couple days are of the brick wall variety, rather than the "blind discovery" variety. I think student teachers are using the second method when we ask them to get some clinical experience. I expect them to discover something on their own when they student teach, yet, it's my job as a teacher trainer to pose a situation where I think the chances are optimal that they will.

Feiman: That is partly why I said that we need to figure out how school experiences should be organized. I think we need to talk about what they are for. It might be good or bad for prospective teachers to have experiences in the classroom. It may also be that, to a great extent, the quality of the experience is a function of the person experiencing it. I think we ask clinical experiences to bear the responsibility for a lot of things that we haven't sorted out sufficiently. Are the student teachers there to find out how to vitalize theoretical principles, to find out what kids are like, to find out what they are like with respect to kids, or just to get acclimatized? Until we sort these things out, time spent in schools is still "blind discovery."

LeBlanc: Admittedly, the variables of a classroom need definition, but let's get back to the problem of content. It would be awfully nice if a first grade teacher knew that he/she was to teach Hebrew to everybody or hunting to everybody. Yet,what are teachers expected to teach? The basic facts? We give them a hard time when they do that. Teachers are doing exactly what they are expected to do and more. Certain teachers do try to make mathematics come alive via social interaction such as basic counting exercises. How many children are here? How many

girls? How many more boys than girls, etc. Where is the developmental idea here? Kindergarten and first-grade teachers need to teach basics. We haven't assigned them very ambitious tasks in the curriculum, let's face it.

Bell: We have the further problem of communicating new tasks to inservice teachers. While we can deal with preservice teachers in our training programs, there are millions of practicing math teachers we can't reach in order to give them a more developmental outlook of math. Perhaps the most effective inservice training for teachers as well as the most effective implementation device is to put teachers in situations where they invent for themselves the things that they themselves will do in the classroom.

I once did a study to find out whether some of the things that everyone is supposed to learn in math,like approximations and orders of magnitude,were practical content for teachers in first, second, and third grade. Our researchers talked to practicing teachers about estimation and approximation in the sense that they were all-pervasive skills, with a kindergarten version, a second-grade version, a third-grade version, etc. Then we asked them, as professionals, to tell us how to implement such a developmental scheme in the curriculum. Their ideas were excellent. The selection of them in my files is extremely rich -- a source for ways to deal with chance, probability, logic, orders of magnitude, estimation, geometry, measure, and a lot of other things that ostensibly kindergarten and first-grade kids are not supposed to learn. Yet, once those teachers knew what we wanted, they knew how to do it. I came to believe from that experience that the most powerful,

most effective way to both implement and train teachers is to engage them in the process of invention, though that isn't the sort of thing I would do with preservice teachers.

Feiman: Bell's example brings to light another important area -- the belief that teachers teach the way they are taught, and therefore we should teach them the way we want them to teach. I think that we haven't really distinguished between how teachers come to understand mathematics and how that understanding of the content helps them decide how to teach it. They need to make distinctions between their own understanding of what it is that they are trying to teach and the decisions they need to make about how they are going to teach it. There may also be differences between how they as adults come to understand mathematics and how they are teaching it to their students. It is one thing to spend half an hour getting the feeling of a concept for yourself and another thing to be able to translate that concept for kids of different ages. I'm very uncomfortable with the idea that "the way I do it with you is the way you are supposed to do with the kids." Perhaps the differences are not so clear before the clinical experience, since Masterson mentioned how student teachers go out to teach and then come back with other questions than those we taught them to deal with.

Scott: I am concerned about the way everyone has been talking about pre-service teachers, but not inservice teachers until this particular session. I don't know what the situation is in other states, but in Michigan we have a supply of teachers for the next 30 years. It seems to me our emphasis must be on inservice training. There are certainly problems unique to that goal. The union is one. They are not going to

let us decide what kind of inservice activities teachers need; they are going to give us guidelines for precisely what they want us to do. They will want it to be relevant to classrooms immediately.

Davis: It is true, teachers are going to come to teacher trainers and say, "this is what we want." The problem is that some of them really don't know. Take an example Bell gave the other day of a second-grade teacher who successfully taught multiplication as repeated addition. He taught it so successfully that the kids became inflexible and had problems in the fifth grade, sixth, seventh grade. That second-grade teacher isn't going to come to you; you need to go to him with something that he wants.

Schwab: The problem is how you sell a teacher something he/she doesn't want to buy.

Davis: The question we should be asking now is how to sell people the things that they need.

Feiman: One of the issues teacher educators have locked horns with the unions over is the issue of teacher centers and workshops. I'm afraid that what is precious and important about the idea of these teacher centers and workshops may well get lost in the shuffle. Essentially, they are set up to create a professional community and context where teachers can be together and share ideas over time. Existing centers rely on self-selected participation, working with those teachers who choose to come. Teachers come because what they find at the centers seems valuable. Working closely with the staff of one such teacher center, I found

that the staff began thinking after a while about ways to stretch the teachers' conceptions and perceptions of needs. One might have to begin by giving people things they are asking for, but that doesn't mean one must stop there. We need to create those settings where dialogue and sharing of experiences can take place -- the sort of activity Bell was describing -- creating an environment for exchange among inservice teachers.

Masterson: USMES (Unified Science and Math for Elementary School) has a program that involves a continual interchange between teacher trainers, curriculum people, and teachers. Working together, the group comes up with a product, tries it out in the schools for a month, and then gets back together to completely rewrite the materials. After two years of this, a book was put together and published in rough form, then the same process was repeated until the product was finalized. I've never heard of any other curriculum program put together in such a unique way.

Bell: That ought to tell us something about effective ways of training teachers, though not about effective ways of curriculum development. You can produce small well-developed centers a lot easier than you can give a program to the whole world.

Walter: Maybe what you are saying is that curriculum development should be part of the training of an elementary school teacher.

Bell: I think it should.

Masterson: What we need to realize is that there needs to be continual training of teacher trainers as well.

Bridgham: I just want to comment on a teacher center that was formed

spontaneously by some historians at the University of Massachusetts. It sounded like a marvelous thing. The teachers explored together what history meant and what it would mean to do the same kind of discovery work in their classrooms. It went on for some time until somebody decided the program was a failure unless they took their ideas and put them into national use. It was like poison in the well. What happened was that this group of people who were interacting and working on ideas for their own classrooms began to worry about a national audience, not about themselves and their own students. The project dried up as both the source of curriculum ideas and as a source of renewal for the teachers.

Feiman: There is a fellow in Cambridge who has been running what he calls "teacher seminars on children thinking." Recently, in a national meeting, somebody raised the question about how the fruits of the seminars could be packaged and disseminated to other people. When the group tried to do this, the project disintegrated. What was for the group an extremely valuable experience became quite transformed in their eyes by the requirement that they somehow package their insights and make them available to a larger audience.

Kilpatrick: Perhaps one of the problems we suffer from in this society is that a person hasn't been successful unless he/she has gone big time. The implication was that USMES, for example, was somehow not successful because it was only used by the people who had been through the program. Americans, I think, are very deeply ingrained with the idea that unless you make a national impact, you have not done anything significant to solve your problems. I think we need to get away from that

in our research, in our teacher education, and in our curriculum development.

Lesh: I too believe that teachers can help develop curricula, but I doubt that that would be an efficient use of their time. I don't want to write my own materials for every course I teach. I want somebody to give me something so I can do what I think is my job. I'm not saying that it's not a good idea to get local input, I'm saying the problem of teacher input is not simple.

Davis: There is an underlying problem here. Teachers have a very arrogant job because they can't really leave it up to the kids to decide what they want to learn. Kids are not going to come into class saying, "I want to learn multiplication," "I want to learn long division," or "I want to learn to spell well." The teacher has to tell them that that is what they want. The teacher must be a dictator.

The presumption at universities is the opposite. If you are teaching complex variables, you must publish in that area. If you are teaching cognitive psychology, you must publish in that area and show that you are qualified to make demands on your students as to what they should learn. Now look what we do with elementary school teachers. We don't ask them to show that they are qualified to make demands, but somebody has to make them. The elementary school teacher is not qualified to say which of various things kids need, just as a math department in which nobody has published for years and years will not be qualified to teach mathematical research. How does the teacher prove that he/she is qualified to make demands on kids? We are really copping out when we say, "let the society decide what it wants." People want big cars, they want to use energy at a rate at which they can't. Somewhere, someone has

got to come out and say people can't have what they want. Indeed, they don't really want what they want.

APPENDIX

PARTICIPANT LIST

Robert Ashlock	University of Maryland
Max Bell	University of Chicago
*Gabriella Belli	Michigan State University
Glenn Berkheimer	Michigan State University
Isobel Blyth	Michigan State University
Robert Bridgham	Michigan State University
Jere Brophy	Michigan State University
Byron Brown	Michigan State University
Joe Byers	Michigan State University
Chris Clark	Michigan State University
Bill Cole	Michigan State University
Calhoun Collier	Michigan State University
Thomas J. Cooney	University of Georgia
Robert Crocker	Memorial University (Newfoundland)
Patricia Davidson	Boston State College
Robert Davis	University of Illinois
Theresa Denman	Detroit (Michigan) Schools
Edward Esty	National Institute of Education
Sharon Feiman	University of Chicago
James Fey	University of Maryland
*Bill Fitzgerald	Michigan State University
Robert Floden	Michigan State University
Thomas Good	University of Missouri
Keith Hamann	University of South Australia
Robert Hill	Michigan State University
Jeremy Kilpatrick	University of Georgia
Lucy Knappan	Lansing (Michigan) Schools
Jacob Kounin	Wayne State University
Leo Landa	Brooklyn, New York
Judith Lanier	Michigan State University
*Perry Lanier	Michigan State University
Glenda Lappan	Michigan State University
John LeBlanc	Indiana University
Richard Lesh	Northwestern University
Jed Lewis	Michigan State University
John Masterson	Michigan State University
Bruce Mitchell	Michigan State University
James Moser	Wisconsin R & D Center
Elizabeth Phillips	Michigan State University
Andrew Porter	Michigan State University
Tom Post	University of Minnesota
Joyce Putnam	Michigan State University
*Linda Raab	Michigan State University
Gerald Rising	State University of New York at Buffalo
Sid Rockland	University of Georgia
Charles Roberts	Michigan State University
Wayne Scott	Michigan State Department of Education
Robert Scrivens	Michigan State University
Lee Shulman	Michigan State University
Richard Shumway	Ohio State University
*Janet Shroyer	Michigan State University
Ed Smith	Michigan State University
Jane Swafford	Northern Michigan University
John Wagner	Michigan State University
Marion Walter	Harvard University
Donna Wanous	Michigan State University
Mary Winter	Michigan State University
Lauren Woodby	Michigan State University

*Steering Committee

Revised Program

Institute for Research on Teaching

MATH CONFERENCE

May 1 - 4, 1977

Michigan State University
East Lansing, Michigan

SUNDAY, MAY 1 -- 6:30 P.M.

Opening Session: Starboard Tack Conference Room

Topic: RESEARCH ON THE TEACHING OF MATH: A CLINICAL INFORMATION PROCESSING PERSPECTIVE

Presider: Bill Fitzgerald
Department of Mathematics
Michigan State University

Presenter: Perry Lanier
Department of Elementary Education and
Institute for Research on Teaching
Michigan State University

7:45 Reception (Cash Bar)

MONDAY, MAY 2 -- 9:00 A.M. - 12:00 NOON

Session I: Erickson Hall, Room 133D

Topic: CURRICULAR ISSUES AND THE IMPLICATIONS FOR RESEARCH ON TEACHING

Presider and Discussion Leader: Andrew Porter
Department of Educational Psychology and
Institute for Research on Teaching
Michigan State University

Presenter: Robert Davis
Curriculum Laboratory
University of Illinois

Reactors: Fred Goodman
School of Education
University of Michigan

Robert Bridgham
Office of Medical Education, Research and Development
and Institute for Research on Teaching
Michigan State University

MONDAY, MAY 2 -- 1:30 - 4:30 P.M.

Session II: Erickson Hall, Room 133D

Topic: LEARNER ISSUES AND THE IMPLICATIONS FOR RESEARCH ON TEACHING

Presider and Discussion Leader: John Masterson
Department of Mathematics
Michigan State University

Presenter: Jeremy Kilpatrick
Department of Mathematics Education
University of Georgia

Reactors: Leo Landa
formerly Professor and Director of the
Laboratory Institute of General and Educational
Psychology
Academy of Pedagogical Sciences
Moscow, U.S.S.R.

Jere Brophy
Teacher Education, Educational Psychology and
Institute for Research on Teaching
Michigan State University

TUESDAY, MAY 3 -- 9:00 A.M. - 12:00 NOON

Session III: Erickson Hall, Room 133D

Topic: ENVIRONMENTAL ISSUES AND THE IMPLICATIONS FOR RESEARCH ON TEACHING

Presider and Discussion Leader: Janet Shroyer
Institute for Research on Teaching
Michigan State University

Presenter: Max Bell
Department of Education
University of Chicago

Reactors: Jacob Kounin
Educational Research Project
Wayne State University

Robert Hill
Lansing (Michigan) School District and
Institute for Research on Teaching
Michigan State University

TUESDAY, MAY 3 -- 1:30 - 4:30 P.M.

Session IV: Erickson Hall, Room 133D

Topic: TEACHER EDUCATION ISSUES AND THE IMPLICATIONS FOR RESEARCH ON TEACHING

Presider and Discussion Leader: Glenda Lappan
Department of Mathematics
Michigan State University

Presenter: John LeBlanc
Mathematics Development Education Center
Indiana University

Reactors: Sharon Feiman
Department of Education
University of Chicago

Judith Lanier
Teacher Education and
Institute for Research on Teaching
Michigan State University

Happy Hour and Catered Dinner ($6 per person) at the home of Jan Shroyer

WEDNESDAY, MAY 4 -- 9:00 A.M. - 12:00 NOON

Closing Session: Erickson Hall, Room 133D

Topic: CONFERENCE SUMMARY

Presider: Perry Lanier
Michigan State University

Presenter: Lee Shulman
Educational Psychology, Medical Education and
Institute for Research on Teaching
Michigan State University

Reactor Panel: Robert Davis
University of Illinois

Jeremy Kilpatrick
University of Georgia

Janet Shroyer
Michigan State University

John LeBlanc
Indiana University